Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation by Laurence E. Sigler

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Fibonacci, also known as Leonardo Pisano or Leonardo Bigollo, was born in 1170, the son of a customs officer. He lived and worked, probably as a merchant, in different parts of the Mediterranean, learning the mathematics concerned with trade and exchange but also Euclid's *Elements*. He came to the attention of the emperor Frederick II of Hohenstaufen, a patron of the arts and sciences who had founded the University of Naples in 1224, and whose court included people like Domenicus Hispanus, an astronomer and astrologer, Theodorus of Antiochia, again an astrologer and a translator from the Arabic, and Michael Scotus, an astrologer, a translator from the Arabic, as well as a philosopher. It was to the latter that Fibonacci dedicated his *Liber abaci*. He also wrote *Practica geometriae* (1220, dedicated to a Domenicus, probably Domenicus Hispanus), *Flos* (around 1225, dedicated to Cardinal Ranieri Capocci), a letter to Theodorus of Antiochia (around 1225), and *Liber quadratorum* (1225, dedicated to Frederick II himself). After extensive travelling, by 1220 Fibonacci seems to have settled in his native Pisa, where in 1228 he was granted a state pension, and where he probably died in 1240.

There is something of a mismatch between Fibonacci's fame and the relative obscurity in which his original works found themselves. Imitated, abstracted and built upon in the mathematical literature ever since the 14th century, his books were nonetheless first printed only in 1838.[1] Historians of mathematics seem to have studied Fibonacci primarily because of his 'anticipations' of later results; at present, there are rather few publications, and fewer still in English,

[1] See Arrighi 1966, 27–29 and 1970; Vogel 1971.

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on the traditions of mathematics with the abacus in the Middle Ages of which he is a central figure. Thus, although many people, even those with no particular mathematical ability, will have heard of the Fibonacci sequence, this volume is the first integral English version of the book where that sequence appears.

There are twelve manuscript copies of the *Liber abaci*, three of which are complete. Sigler's translation follows the Latin text edited by Baldassarre Boncompagni, which is based on one manuscript.[2] The incipit of this latter simply bears the date 1202, but other manuscripts specify that the work was first written in 1202 and then corrected in 1228. Fibonacci himself, addressing the 'most great philosopher' Michael Scotus in the dedication of the *Liber abaci*, mentions that he had already sent him a book on numbers [15]. Boncompagni's and Sigler's text must correspond to the 1228 edition, because in it Fibonacci refers to the *Practica geometriae* [15] and the *Liber quadratorum* [261]. The work comprises fifteen chapters, starting with a dedication and prologue where Fibonacci gives some autobiographical details, and insists on the interconnection of geometry and arithmetic on the one hand and of theory and practice on the other. He states that he intends to combine the former two, by providing 'many proofs and demonstrations which are made with geometric figures' [15] and by adding to the 'Indian method' others taken 'from the subtle Euclidean geometric art' [16]. As for theory and practice, Fibonacci declares that the *Liber abaci* in fact 'looks more to theory than to practice' [15].

The unique selling point of the book is its introduction of the 'nine Indian figures' to a more general public, and in particular to the Italians [16]. Indeed, chapter 1 starts by explaining the use of the nine figures, plus the zero, which Fibonacci calls *zephir* (*zephirum*) following the Arabic. These figures are favorably compared with traditional Roman numerals in order to understand place value and use of the *zephir*; the reader is also reminded of the finger signs for numbers, 'a most wise invention of antiquity' [20], according to which, for instance, curving the middle finger makes 5, curving the forefinger over the curve of the thumb makes 60, and so on.

[2] See Boncompagni 1857–1862, vol. 1.

Chapter 2 is on the multiplication of integers and includes methods to check whether the result is correct (what we today call 'algorithm'). Multiplication, and later division, require the 'keeping in hand' of numbers (today's 'carrying'); both come across as very physical operations involving memory, writing, and the fingers (which function as an extension of memory). In the dedication, Fibonacci had said that memory, intellect, and habit must work together with hands and fingers instantaneously, as if 'with one impulse and breath' [15]. Chapters 3 and 4, on addition and subtraction respectively, also provide methods for checking whether the calculations are correct. Chapter 5, on the division of integers, includes tables of division up to 13 and introduces irregular numbers 'for which no rule [of composition, i.e., division into factors] is found' [69]. Chapter 6 deals with multiplication, this time of integers with fractions (*rupti*), which in chapter 7 are added to one another and subtracted. Chapter 8 starts a sequence of practical problems: finding the value of merchandise [ch. 8], the barter of merchandise [ch. 9], companies [ch. 10], alloying [ch. 11], 'problems of abaci' in general [ch. 12, the longest], the *elchataym* method (or method of double false position [ch. 13]), roots [ch. 14], geometric rules and problems of algebra and *almuchabala* (from al-Khwarizmi's *al-Jabr'w'al muqabala*, i.e., rules of restoration and reduction [ch. 15]).

The *Liber abaci* is a veritable treasure chest not just for the historian of mathematics or science but also for the historian of medieval economy and society as well as for the scholar interested in 'East-West' relations during the Middle Ages. There is, as it were, something for everyone. On the more technical side, Fibonacci's arsenal of solution procedures is particularly remarkable. Apart from methods based on a largely Euclidean proportion theory, we find false position, direct method ('used by the Arabs,... a laudable and valuable method' [291]), indirect method (a sort of inversion of the direct method, which also employs an unknown called 'the thing'), and double false position or *elchataym*, which Fibonacci variously presents as that 'by which the solutions to nearly all problems are found' [447], as 'necessary' even when it is not ordinarily considered [466] and as 'miraculous' [477]. Chapters 12 to 15 will give ample food for thought to those interested in geometrical algebra and in the developments of Euclid's *Elements* book 10. In fact, 'the most skilful' [57], 'most illustrious geometer Euclid' [107] is the main authority cited by Fibonacci. He also mentions Ptolemy and the *Almagest* [180], Ametus the Younger [180], 'a certain Constantinople master' [28], and (but this is a note on the margins of the manuscript) 'Maumeht', i.e., Mohammed ibn Musa al-Khwarizmi [554]. As is well known, Fibonacci was probably taught by Islamic teachers in North Africa; he identifies different mathematical traditions—Arab, Greek, Indian—and sees his work as a combination of them [16].

As stated in the dedication, the book also combines theory and practice, *scientia* and *ars*. At the beginning Fibonacci refers to the subject at hand as a *scientia* [15], yet throughout the book he talks of *ars*. The *scientia* in question is in effect profoundly practical because it has to be achieved through exercise, with a combination of habit, memory, and intellect in accordance with hands and figures [15]. On the other hand, the imperfections typical of an art are present: some solutions can only be approximate or found 'God willing' [526], many of the methods entail angling for the correct answer through (educated) guesses; the expert gets a feeling for the problem and sometimes does what 'looks good to [him]' [369] rather than following a strict procedure. Given that not all problems are solvable, or that some of them in some cases would produce irrational or negative solutions, whenever possible Fibonacci trains his reader to recognize solvability or insolubility by simple empirical tricks [e.g., 294, 303, 336, 365].

The ways in which Fibonacci's account is made persuasive again reflect this combination. He does not *prove* his results in the axiomatic/deductive sense of the word. Occasionally, he provides geometrical proofs where numbers are translated as lines, and which are Euclidean in style or at least inspired by Euclid. This is evident particularly in chapter 14 where he states that 'according to geometry, and not arithmetic, the measure of any root of any number is found' [491], and in chapter 15 where some old problems return to be tackled geometrically or at least with the accompaniment of little explanatory/demonstrative diagrams [545 f.]. On a closer look, however, it could be argued that the constant repetition and checking of the methods, in evidence from chapter 2 onwards, also constitutes their demonstration, their being evidently valid. At times, Fibonacci says, 'as is demonstrated in the written illustration' [78] or 'as is displayed in this description' [132], referring to nothing more than a written operation, where, if the reader has followed each step, he cannot but agree that the result is as indicated. The concrete example adds clarity to the general rule and is a crucial part of the demonstration [500].

The book contains tables, illustrations, and, when Fibonacci gives geometrical demonstrations, simple diagrams. The illustrations (*descriptiones*), which show the reader how a certain operation is written down, are important because part of the instruction provided consists in keeping things tidy. Operating with the Indian figures in a correct and efficient manner involves putting a certain figure at a certain step in the calculation in a certain position, above or below another figure. The organization of the small space enclosed by the illustration is paramount for the solution of the problem at hand. Again, some methods (such as the rule of six proportionals, [184]) require a careful arrangement of known and unknown quantities along upper and lower lines.

Indeed, Fibonacci has these words of advice for the learner, to quote in full what we have mentioned earlier:

> [he] ought eagerly to busy himself with continuous use and enduring exercise in practice, for science by practice turns into habit; memory and even perception correlate with the hands and figures, which as an impulse and breath in one and the same instant, almost the same, go naturally together for all; and thus will be made a student of habit. [15]

The chapters that follow bear this out in their relentless sequence of exercise after exercise. When dealing with elementary operations, a rule is applied to a concrete example from its very introduction; more concrete examples follow, sometimes in a crescendo where, for instance, the multiplication is first of a two-figure number by a two-figure number, then three figures by three figures, then four figures by four, and so on. Fibonacci accompanies the reader through most of the steps (he only starts skipping steps after a couple of examples of a certain method have been provided), occasionally explains why a step produces a result [125], and every now and then repeats the general set of instructions (do this, put the figure there), as if literally to drill it into the learner's mind. There are references to the care needed to carry out calculations without mistakes; and he not only provides rules for checking both the calculations and the solutions to some problems, he insists upon these checking rules almost to the same degree as the rules and methods for making the original calculations. Fibonacci expects his reader to have a good memory, and to retain the contents of most of the book's tables and the main procedure of most of its paradigmatic problems [211] by heart. Particular procedures are made memorable by constructing a little story around them: we have the problem of the tree, that of the purse, that of the man travelling from city to city, and Fibonacci, having provided three or four examples for each problem, can later refer to, e.g., 'the same method as in the tree problem' when a similar procedure is required [e.g., 252, 255, 396, 438].

Sometimes it looks as if Fibonacci wants to go in the direction of greater abstraction: while dealing with the problem of 'horses that eat barley in a number of days', he denotes the numbers in question with letters, before providing a general rule on how the problem is to be solved [206–207], and he does the same throughout chapter 14. There are problems about numbers in themselves, rather than about numbers as attached to specific things [259 ff., 310 ff., 316 ff., 431–433]; but even then on one occasion he specifies 'the rules for the summing of series were indeed shown; now truly applications of them are shown . . . There are two men who propose to go on a long journey . . . ' [261]. Indeed, even the most 'abstract' chapter, ch. 15, applies some of the general, geometrically-demonstrated rules to concrete money problems akin to those of chapters 8 to 12 [541, 557, 564].

Fibonacci states clearly that his account has a practical aim, and can be useful for business [120]; he helps the reader to avoid labor in calculations by providing shortcuts [153]; he even deals with the minting of coins with a certain content of silver and copper [233], and concludes

> Indeed from this rule follows a certain valid pattern often useful in this method of monies. Indeed the money that is made sometimes comes out with an excess, sometimes with a deficit, that is sometimes with too much silver, sometimes too little silver; sometimes it is too weak because of lack of knowledge in alloying, or the copper is deficient or excessive because of boiling. [239–240]

The insights into the world of international trading in the 13th century are numerous and invaluable. Objects of calculation include pepper (a 'not very expensive merchandise' [163]), cloth, hides, cheese of different qualities, saffron ('expensive merchandise' [163]), nutmeg, oil of Constantinople, sugar, pork, rabbits, birds, alum, mastic, cinnabar, and false silver (silver mixed with tin); currencies exchanged range from pounds of various kinds to massamutini to bezants; and the units of measure whose relative proportions and equivalences are found come from Pisa, Provence, Palermo, Messina, Cyprus, Syria, Alexandria, Genoa, Turin, Florence, Barcelona, Padua, Bologna, Venice, Tarentum, and Barbary, the coastal area of North Africa (where Fibonacci had learnt about the Indian figures and their method). With such metrological variety, one 'must do with all things according to the diversity of weight and parts of them, and according to the custom and order of the provinces in which you will have to operate' [163]. We are also told *en passant* of exchange surcharges, of duty tolls, of commissions on commercial transactions that take place on certain markets, of coins and their value (in some cases dependent merely on their silver content, which can be determined by melting them), of banks ('houses') and interests, and of various types of associations for profit. A merchant woman makes an appearance as a seller of apples and pears [250]; little stories are told of workmen who lose almost all their salary to their employers or foremen on maintenance or sickness [392, 453], or of soldiers who acquiesce to unfavorable terms for the payment for their fiefs because the terms are set by kings [392].

On at least one occasion, the rule that Fibonacci proposes may be derived from actual contemporary practice:

> [T]his method is much used in the loading of ships when diverse merchandise is loaded, and is had according to the diversity of weight, the lightness or heaviness of them, as when the ships that are loaded in Barbary, and are filled with loads of hides. [176]

The rules about reductions of weight are conventional, may vary from place to place, and date in some instances from ancient times; and 'certain of these we propose the use of in this work' [176].

The question of what was the intended audience of the *Liber abaci* merits further attention. It seems to be directed not only to merchants and their sons, but also to the court. Some of the problems evoke leisurely scenarios: there are party games involving guessing a number ('if you will tell him that he [is thinking of] 27 you will see this called a miracle' [435]) or people sitting together and hiding a ring [430], and fable-like stories of a lion in a pit (which takes 1575 days to get out [273]), two serpents one at the bottom and one at the top of a 100-palms-high tower [274], a dog chasing a fox [276]. Fibonacci is interested in effect: there are several references to 'elegance', or to an expression being more elegant than another [e.g., 81, 194]. There is also an 'optimal' way of arranging parts of a fraction so that checks are easier to carry out and the fraction looks less unwieldy. Once we are treated to Fibonacci's humor: a merchant carries precious stones to Constantinople, passing by three custom houses. The first custom agent remits his fee because they are friends. The other two do not accept the remittance. Fibonacci then rejoins 'that which was said of the first custom house is said only in jest to impede the untutored' [396]. My personal favorite in its almost *A Hundred and One Nights* evocation of secluded orchards, demanding custodians, and the eventual punishment of greed, is the problem of the man who entered a pleasure garden through seven doors, took a number of apples, went back and lost all the apples but one to the seven doorkeepers, who one after the other claimed some for themselves [397]. Having asked how many apples the man collected in the first place, Fibonacci starts from the last solitary apple and constructs the series of 'confiscations' backwards, leading to the original amount. The rabbit problem, containing the now-famous Fibonacci sequence 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, representing the monthly generations of rabbits springing from one initial couple, is a few pages later [404].

Laurence Sigler, who was also the translator of Fibonacci's *Book of Squares* into English, unfortunately died before seeing this volume through the press. His translation, by all accounts a huge undertaking, reads fluidly enough; and he does justice to the original in not skipping passages for the sake of avoiding repetitions, and in resisting any temptation to 'update' the text or to number the propositions, which are distinguished only by their subheadings. He also thankfully eschews a 'modernizing' stance by reserving references to Fibonacci's successors and recastings of his results into contemporary notation to the endnotes (e.g., p. 619, for Fibonacci's 'anticipating' Gauss' theory of residues). There are a few minor missteps in rendering the Latin; the English text contains some typos, including in the numbers and the bibliography; some of the diagrams in books 10, 12 and 13 have been modified with respect to the Latin version. Nonetheless, those are minor flaws in a publication that will hopefully make big waves and open the world of Leonardo Pisano to new generations of readers.

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