Plato's Ghost: The Modernist Transformation of Mathematics by Jeremy Gray

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The period from 1870 to 1930 was a time of significant cultural change. In art, literature, architecture, and music, these changes led to new ways of doing things that went under the name of 'modernism'. Jeremy Gray's central argument in *Plato's Ghost* is that the same concept may be usefully applied to the history of mathematics in this period.

Modernism arose in reaction to Late Romanticism, in which previous ideas and techniques seemed to have reached their limits. Realism in literature could not, it seemed to many, be pushed any further than it had been by Zola. The flirtation with fantasy and fairy tales during the Victorian era also seemed not to lead anywhere. In art, Impressionism had called everyone's attention to the role of the perceiving eye as mediator between reality and the human observer. Wagner's music was felt to have pushed emotional intensity as far as it could go.

Science and philosophy also contributed to the unsettling of the old ways. Once the Christian consensus fell apart, the foundations of Western thought began to seem shaky. Philosophers began to wonder how to justify our conviction that we know anything about the world. Marx questioned the whole economic structure of Western civilization in a way that seemed, if not persuasive, at least worthy of consideration. Darwin's discovery of evidence for evolution led to a revolution in biology. Physicists dealing with electromagnetism ran into more and more difficulties. Finally, technology was becoming a bigger part of the everyday life of most Europeans and Americans,

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changing the way that people interacted with their physical environment and thereby removing from nature the mystical aura that had so enthralled the Romantics.

The modernists called into question the whole structure of the culture that they had inherited, highlighting the tensions. They questioned the link between art and reality, and the idea that reality trumps human perception and imagination. Houses designed for no other purpose than to be lived in began to be admired as objects in and of themselves. Music migrated from homes to concert halls, art moved to museums, and it became fashionable to speak negatively of 'mass culture'. The modernists seemed unconvinced by the naïve progressivism of the 19th century. The deep past interested them more than recent times, but they wanted to preserve this legacy only by embedding it into a completely new framework.

The modernists were acutely self-conscious, probably more so than any other cultural and artistic movement. This was the time of manifestos, of schools and programs, of the proliferation of new 'isms'. More people than ever before learned to read and to appreciate art and music, but popular approval became identified with lack of quality. The cultural products of the modernist school seemed to be aimed at a select few, to those who understood their ideological and artistic background, to those who could appreciate the technical difficulty of much modernist art. To appreciate the new styles, one also needed to be aware of the past, to be able to understand the quotations and ironic misquotations of past works that filled the new art.

First art, then also music, literature, and architecture became abstract. Rather than continue to use longstanding conventions about how the perceiving subject interacts with cultural products, the modernists produced art that broke all the rules. The new cultural products seemed to say 'Forget your expectations, take me or leave me as I am.'

By the 1920s, modernism dominated the world of art. In literature, the new approach never quite achieved that kind of unanimity, especially in the realm of the novel, where new and old continued to survive and in fact to influence each other. Modernist music faced (and still faces) an uphill battle, given that the public for the most part refused to go along. Modernist architecture won the day when it came to monumental (and often government-funded) buildings, but most people's houses remained much as they always had been. One way or the other, by the 1930s modernism was the establishment.

This account of the modernist moment is, of course, vastly oversimplified. Historians still bicker about the details, but most seem to agree that 'modernism' is a useful way of understanding the period. The transformation that hit several kinds of artistic and cultural production during this period was real, and the changes were sufficiently similar in spirit that they all deserve a common name. There have been many attempts to describe what exactly 'modernism' consists of; most would probably agree with the following:

- Modernists were much less concerned about the connection between their art and external reality than the artists before them.
- There was a break with traditional forms (verse became 'free'; music, atonal; novels, experimental; and art abandoned the goal of being beautiful).
- The greatest achievements of the past were not abandoned, but they were systematically reinterpreted, reclaimed, and transformed.
- An acute self-consciousness led to aggressive agendas and manifestos.
- Artists wanted the support of the masses and of governments, but felt that only the truly educated had any right to pass judgment on their work.
- Form (which includes deliberate lack of form) and technique became more important than content. The idea that artists could be called to task for what they said was viewed as repressive.

No one doubts that mathematics (and physics) went through an equally dramatic transformation between 1870 and 1930. This was the time of the popularization of non-Euclidean geometries, of Relativity and Quantum Theory, of debates about the foundations and ultimate reliability of mathematics. This period saw the advent and triumph of abstraction as a mathematical technique, with new attention being paid to logic and axiomatics. It was also the time when mathematics became a profession, when universities began to focus on research and professional societies were formed.

One can also point to a kind of 'late Romantic' crisis in mathematics. What was one to make of the surfeit of formulas to be found in the work of Kummer and Jacobi? (The impenetrable first chapters of Gray's book on *Linear Differential Equations and Group*

Theory serve as a good example of this problem.) Things seemed to be getting so complicated that progress required a completely new approach. This helped lead to the new mathematics of the late 19th and early 20th centuries.

At the core of *Plato's Ghost* is the thesis that these two transformations can be usefully correlated. Between 1870 and 1930, mathematics became dramatically more abstract, more concerned with 'structure' than with specific examples of phenomena, and much less connected to the sciences. Abstraction required standards of proof that were new and much more formal than ever before. Mathematicians ceased to worry about the approval of society or even of scientists, creating their own journals and their own standards for what constitutes good work. And, as never before, they argued about foundational issues, about what was acceptable mathematics and what was not, about the relationship between mathematical truth and the real world.

The nature of the transformation at issue can be illustrated by a famous anecdote. In his study of integral equations, David Hilbert formulated a notion of proximity for functions that allowed him to make sense of the idea of producing solutions by approximation. Mathematicians more modern (or more modernist) than Hilbert took this notion and formalized it, creating a concept they called 'Hilbert space'. The anecdote tells of Hilbert attending a seminar in which the speaker began talking about a certain Hilbert space. The eminent mathematician whispered to a nearby colleague, 'Can you tell me what a Hilbert space is?'

It is precisely this sense of the old made new, of *ad hoc* techniques being turned into formal structures, of theory triumphant over problem solving, that we see throughout mathematics in these decades. Talk about 'objects' and 'spaces' replaced talk about 'formulas' and 'equations'. There was an almost total transformation of algebra, as anyone who compares the tables of contents of, say, Perron's *Algebra* (1927) and van der Waerden's *Moderne Algebra* (1930) cannot fail to note. (In fact, the adjective 'modern' became firmly attached to the new approach to algebra, and continues to be used in this sense.)

It would have been inconceivable for a mid-19th century mathematician (say, Bernhard Riemann) to have the attitude toward applied mathematics that we find in G. H. Hardy's *A Mathematician's*

Apology. (This despite the fact that Hardy's mathematics is hardly 'modernist'!) For Riemann, mathematics and physics were tightly linked, and the idea that one should (or even could) do mathematics without any concerns about what is 'useful' would probably have struck him as bizarre.

To most mathematicians today, the 19th century arguments about geometry seem equally strange. The discovery of non-Euclidean geometry generated a passionate argument about what the 'true' geometry was. For the scholars in question, geometry was about describing the real world, the three-dimensional space that we perceive. Today, that seems absurd. For us, geometry is an abstract bit of mathematics like any other, and we can prove theorems about the geometry at hand without worrying whether it corresponds to anything in the real world. When it comes to applying all this to reality, one simply chooses the most convenient geometry for the problem at hand. And just as we find them hard to understand, they would probably be puzzled by our talk of 'a space' or, worse, of 'spaces'.

Of course, that does not mean that mathematicians feel that they are playing an abstract game with no intrinsic rules. However abstract an object the Riemann zeta function may be, most mathematicians will argue that its value at $s = 3$ is a *specific* number, whose properties (e.g., is it a fraction?) are *investigated* rather than created. Philosophically, this seems very problematic. What kind of reality is there to a number that can only be specified *via* a complicated (and potentially infinite) convergence process? Do we 'have' a number if we cannot (even in principle!) do more than produce approximations to it? When one talks about that number, is one talking about one specific *thing* or about the approximation process itself? Given that mathematics seemed more and more loosely connected to science, the question of what warrant is available for mathematical 'facts' presents itself very forcefully.

Today, most mathematicians are happy to leave such questions to the professional philosophers. (I suspect that, as Henri Lebesgue suggested long ago, the philosophers are grateful for this.) In the early 20th century, however, mathematicians were deeply involved in such arguments. More remarkably still, they allowed such questions (and their proposed answers) to affect their practice of mathematics. The most dramatic example of this is L. E. J. Brouwer, who proposed a radical purge: infinitary arguments should simply be abandoned. The crisis provoked by this proposal plays a large role in *Plato's Ghost*, reflecting the large role it played at the time. In fact, Gray's discussion of philosophical issues is much more extensive and detailed than most mathematicians today (I include myself) can really stomach. In fact, such philosophical questions at times threaten to take over the argument. It was that way in the 1920s as well.

One could easily multiply examples of the difference between the mathematics of the mid-19th and of the 20th century. Of course, there are also counterexamples, including some areas of mathematics where fairly traditional work continued to be done. Gray cites differential equations, for example: in this field one sees *both* very concrete and traditional work and fancy reinterpretations in terms of linear operators, sheaves, and D-modules. There are many other examples, but most of them are similar, with both 'modernist' and 'classical' work happening side by side. (Beyond the purview of this book, one might note that in recent decades a move toward concreteness may have begun, spurred in part by the computer revolution.)

It is clear, then, that there was a transformation of mathematics during this period. It is also clear that the new mathematics shares many characteristics with artistic modernism: it is more abstract, less concerned with 'reality', more formal, harder to learn and appreciate. One even sees echoes of the modernists' rather tiresome fondness for manifestos and arguments about the nature of what they were doing.

Historians have, of course, long been aware of many of these changes. Some, for example, have emphasized the professionalization of the field: the establishment of 'mathematician' as a specific identity, the creation of mathematical journals and national mathematical societies, the rise of the research seminar and of the Ph.D. as the required certification. Gray's description of the new mathematics as 'modernist' is an attempt at a new understanding—and perhaps also a new explanation—of these changes.

In order for the notion of 'mathematical modernism' to be useful, however, one needs a far deeper analysis. It is necessary to examine the period carefully to see exactly what changed and how, in order to be sure that we are not selecting our evidence to fit our thesis. If 'modernist' were just a period label to be attached to whatever we find in the mathematics of the time, then very little would have been achieved. One hopes, in fact, for more: not only must the details fit the overall picture; the new concept of 'mathematical modernism' should also shed new light on the conceptual changes in the mathematics of the period. *Plato's Ghost* attempts to provide such analysis and to argue for its clarifying value. Jeremy Gray is admirably suited for such a task. Over the last decades, he has put together a substantial body of work on the history of mathematics in the 19th and early 20th centuries, focusing especially on geometry, mathematical physics, and the philosophy of mathematics. He has written less on analysis, algebra, and number theory; but he has read widely. *Plato's Ghost* is in many ways a summing-up.

Gray has chosen to address his book to historians of science in general; in particular, he tries not to require of his readers the kind of knowledge of mathematics that a professional mathematician would have. This is probably the right choice, given that mathematicians tend to be interested only in a utilitarian sort of history, in history of the kind that sheds some light on current mathematics. Since Gray wants to argue an eminently *historical* thesis, it is to historians that he directs himself.

This choice does, however, have its costs. The central one is easily grasped: in order to argue that there was a fundamental change in *mathematics* (rather than, say, in the practice of mathematics, or the philosophy of mathematics) one must look at mathematics itself. Since by 1870 mathematics was already very much a specialist topic, this creates real difficulties for non-mathematicians. Gray deals with this issue in three ways. First, he tries to explain some (in general well chosen) mathematical questions of the time and to give the reader some idea of the way they were dealt with. Second, he gives a lot of attention to mathematicians' writings *about* mathematics. Finally, he spends a great deal of time on topics that his readers may have some knowledge of (mathematical physics, non-Euclidean geometry, logic, language) and much less on the more abstract reaches of mathematics (algebra and topology, for example).

The first decision is commendable, the second a little worrying, the third something to be regretted. It is frustrating to see Gray avoid precisely those topics, especially algebra, which are the best examples of the transformation that Gray seeks to understand and document. The deliberate and self-conscious way in which Emil Artin and Emmy Noether remade algebra in the 1920s fits very well into the category of modernism. It is a pity, then, not to read more about them. Gray does give some attention to Dedekind's creation of 'ideals' and to Hensel's p-adic numbers, but those do not really represent the full blooming of modernist algebra. Hensel, in particular, was a transitional figure who does not seem to have ever really adopted (or understood) the newer style.

It is inevitable that Gray would focus on writing *about* mathematics in this period, both as a way of keeping non-mathematician readers on board and because this was a particularly fertile period for such writing. But it does raise questions, in particular the question of the relation between theory and practice. Gray himself notes, in his discussion of David Hilbert, that his mathematical practice was mostly rather traditional while his *philosophy* of mathematics was quite modern. There are other examples in which it seems that the theory is *post hoc*, intended to justify an approach to mathematics chosen, perhaps, for other reasons. Hermann Weyl is an unusually reflective example of this: after flirting with Brouwer's radical intuitionism, he abandoned it because it turned out that one needed infinitary arguments to do quantum theory.

It may be that philosophers will appreciate Gray's detailed attention to these writings. I will admit that I found some of the discussion either boring or irrelevant. Do discussions about language and linguistics (and the late 19th century fascination with artificial languages) really matter to a history of mathematics? Overall, however, one is impressed by the deep knowledge on display and the thoroughness with which Gray surveys the scene. He has convinced me that speaking of 'modernist mathematics' is more than a *façon de parler*, that it can be a useful way of thinking about the transformation of mathematics in these decades.

Gray is particularly strong on the connection between the new mathematical modernism and the foundational crises that shook the philosophy of mathematics in the period. Once geometry was no longer about physical space and mathematics was unmoored from physics, it became important to explain where the reliability (if any!) of mathematical results came from. The naïve answer was a form of

Platonism: there was a mathematical reality ‘out there’ that mathematicians can investigate. Such a position is hard to justify philosophically (especially for non-theists), and perhaps it was this that stimulated the new philosophical currents of the time. Some attempted to reduce mathematics to logic, others decided to discard all the mathematics that seemed philosophically dubious, still others attempted to base the potentially dubious part of mathematics on arguments using only what was universally accepted. Hilbert, for example, wanted to find finite intuitively acceptable arguments that would establish that the mathematics of infinity was free from contradictions and, hence, should be accepted.

As Gray explains, this project turned out to be unsuccessful. After an explosion of interest in such issues, their widespread discussion died out. The philosophical questions were left to the philosophers, and mathematicians mostly went back (unless pressed) to their naïve Platonism. Gray argues persuasively that these changes in mathematical epistemology are directly connected to the modernist project.

Gray has laid out the outline of a research program in this book. He has sketched out the landscape; but, as he says in the first chapter, there are many issues still to be addressed. The most obvious question, which Gray explicitly declines to discuss, is about the cause of the similarities between modernism in the arts and mathematical modernism. Was there influence of one on the other? (William Everdell, for example, includes Dedekind and Cantor among his *First Moderns*, which, given the chronology, puts them at the origin of modernism.) Or was it that the cultural conditions that lead to one also lead to the other? An initial step towards such an investigation would be to try to find out whether the crucial ‘modernist mathematicians’ were interested in the arts, and in what way. One might ask the same about the artists.

Other issues, perhaps more accessible, beg for investigation. How aware were these mathematicians of the ‘modernist’ character of their mathematics? Did they feel that the new approach was the inevitable way to proceed, or was there a conscious radicalism in play?

I expect to see many papers investigating how the transformation played itself out in particular areas. Algebraic topology and probability theory should be investigated. I would be especially interested in a close look at what happened in algebra. Also worth investigating are the pockets of resistance, such as the theory of differential equations and combinatorics. One might also look at individuals. Benoît Mandelbrot, for example, has always presented himself as a kind of anti-modernist; but is the claim justified? It is characteristic of good ideas that they are fertile in this way.

What does all this have to do with Plato and his ghost? The reference is to Yeats' 'What Then?' which appears as an epigraph to the book. In the poem, Plato's ghost repeatedly calls into question the achievements of the main character. Gray's introduction refers this to his own work; but, of course, it also points to the Platonist implications of modernist mathematics and to the unfinished task of sorting them out.

There have not been very many historians of mathematics willing to hazard overarching historical theses. Jeremy Gray is to be commended for having taken the risk. *Plato's Ghost* is an impressive achievement; I expect it to become a touchstone for future research on this period and to bear many fascinating children.