Apollonius de Pergé, Coniques. *Tome 4: Livres VI et VII. Commentaire historique et mathématique. Édition et traduction du texte arabe* by Roshdi Rashed

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In memoriam Hélène Bellosta

With the publication of the fourth volume of the new edition of the *Conics* by Apollonius of Perga (*ca* 262–180 BC), Roshdi Rashed has completed his very important work on the edition of the Arabic text, its translation into French, and a vast mathematical commentary. Apollonius' treatise itself may well be considered one of the highest achievements of Greek mathematics at its most brilliant. In fact, together with the corpus of the mathematical work of Archimedes (287–216 BC), the *Conics* constitute the greater part of Greek *higher* mathematics.

Rashed's edition of the text of the *Conics* is the latest episode in the long and intriguing history of the transmission of this major mathematical work to us. The first four books arrived to Western mathematical culture through the edition by Eutocius (fifth century AD), which was translated into Latin in the 16th century by Johannes Baptista Memus, Francesco Maurolico, and Federico Commandino. Books 5, 6, and 7 of the *Conics* arrived in Europe only through the Arabic translations of the Greek text: the first text of the lost Greek books was contained in an Arabic compendium of the *Conics* written by Abu'l-Fath Mahmud al-Isfahani (second half of the 10th century). This text was given to Cardinal Ferdinando I de' Medici (later grand duke of Tuscany) by the Patriarch of Antioch as early as 1578 but was edited and translated by the Maronite deacon Abraham Ecchellensis (Ibrahim al-Haqilani) under the supervision of Giovanni

Alfonso Borelli only in 1661. This text was heavily manipulated by its Arabic editor.

In the ninth century, the Banū Mūsā brothers (Muhammad, Ahmad and al-Hasan) made great efforts to acquire, understand, and obtain a translation into Arabic of the complete text of the *Conics*. Books 1–4 were translated under the supervision of Ahmad by Hilal b. abi Hilal al Himsi, and books 5–7 by Thabit b. Qurra. (Book 8 is now considered to have been lost by this date). This Arabic translation was brought to Holland by Jacobus Golius in 1629 (it now is in the Bodleian Library in Oxford); but even though its existence was well known in Europe, it was published in a Latin translation by Edmund Halley only in 1710. Halley's edition remained the main reference for books 5-7 of *Conics* until recently and constituted the basis for the first English translation of these books [Heath 1896] as well as of the first French translation [ver Eecke 1923]. A more recent English translation (with the Arabic text) of books 5–7 was published in 1990 by Gerald J. Toomer.

It is worth noting that from Halley's edition on, the Banū Mūsā version has been used to give us the translation of the last three books only of the *Conics*, while the first four books were always published on the basis of the edition of Eutocius directly from the Greek text. As Rashed pointed out, this reveals some prejudices, among which I may cite:

- the idea that the edition by Eutocius provides us with Apollonius' exact text of the first four books of the *Conics*, and
- the idea that the Arabic translation of the first four books is that of this same edition by Eutocius.

The complete edition of the *entire* corpus of the Banū Mūsā version allows us to understand, for example, that there are many differences—and sometimes very profound ones—between the edition of Eutocius and the Arabic translation, mainly in book 4. Rashed points out some of these differences:

- in Eutocius's edition, book 4 consists of 57 propositions but there are only 53 in the Arabic translation;
- some propositions of Eutocius's edition are missing from the Arabic translation;[1]

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[1] An attentive examination of these propositions shows that they may be

- there are two propositions in the Arabic translation that do not appear in Eutocius's edition;
- the order of propositions differs;
- the figures and their letters differ in a certain number of propositions; and
- there are different proofs and, moreover, some proofs are erroneous.

In any case, the Banū Mūsā 's edition of books 5–7 has always been reputed to be the principal source for that part of Apollonius' work and Rashed's edition makes a very important contribution to our knowledge of it.

Book 6 is concerned with the problem of defining equality and similarity between conic sections. The first part (up to proposition 27) treats what we can call the 'criteria for equality and similarity'. The second part (up to proposition 33) poses the main problem: how to cut a given right cone so that the result is a section equal (or similar) to another given one.

This poses an interesting conceptual problem: 'What is really meant by the terms "equality" and "similarity" between conic sections?' Rashed's commentary dedicates many pages to this matter. With regards to equality, Apollonius resorts to the idea of 'superposition': two conics are equal if they can be superposed on one another (by means of a motion). In the words of Apollonius as rendered in Rashed's translation,

> *les sections de cônes que l'on dit égales sont celles dont les unes peuvent se superposer aux autres et dont aucune n'excède l'autre.* [90]

Toomer [1990, 1.264] uses the term 'can be fitted' for Rashed's 'peuvent se superposer'. In any case, this is a usage that goes far beyond what was done by Euclid, who in his fourth common notion effectively says, 'Things which coincide with one another are equal to one another' [Heath 1956, 1.153]. The phrase written in Greek is «καὶ τὰ ἐφαρμόζοντα ἐπ᾽ ἀλλήλα ἴσα ἀλλήλοις ἐστίν». The word «ἐφαρμόζοντα»

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defective. For example, proposition 4.7 depends on a hypothesis which is supposed to have been given in the preceding proposition 4.6; but this hypothesis does not exist.

was translated by Heath as 'coinciding'. The translation of al-Hajjai, reproduced by Rashed, is:

> *Celles (les choses) qui se superposent les unes aux les autres sont égales les unes aux autres.* [10]

While in the Euclidean definition we can thus discern the use of the term 'superposition', it is not at all clear how this is connected to the idea of *motion*. How is this common notion (note that in Euclid this is not a definition) to be used concretely? The absence of any postulate regarding the use of this notion and, in particular, the notion of the rigid motion that would lead the two figures to be superposed on one another, makes verifying the congruence of the two figures problematic.

Actually, Euclid prefers to make use of it in a very limited way. In the second proposition of the *Elements*, he constructs a segment equal to a given segment (thereby showing how to 'move' a segment) but he does not make use of congruence, and the verification of the equality of the two segments is entrusted to postulate 3 ('All radii of a given circle are equal to each other') and to common notions 2 and 3 ('The addition/subtraction of equals to/from equals result in equals'). In the fourth and eighth propositions (criteria for the equality of triangles), he actually uses equality by superposition: from that point on, as far as the equality of polygons is concerned, no use at all is made of superposition. It is another matter with regard to the equality of arcs of circles, whose verification often requires reasoning based precisely on equality by superposition of figures since one cannot resort to equality between triangles as in the case of polygons.

The relevance and the meaning of this definition and its connection with Euclid's common notion has also been discussed by Fried and Unguru [2001] in great detail; and although it might be worthwhile to compare their point of view with that of Rashed, this is beyond the scope of the present review. Still, it is perhaps worthwhile to underline, as Rashed does, the fact that from the point of view of modern criticism, in the absence of any postulate regarding rigid motion, Euclid's reasoning does not appear to be rigorous. On the other hand, Hilbert showed that it is necessary to assume Euclid's

proposition 4 (the first criterion of equality of triangles) as a postulate, given that it is not at all a logical consequence of the Greek mathematician's postulates, axioms, and common notions.

Naturally, Apollonius, who is comparing segments of a conic, is forced to resort more than once to the criterion of equality by superposition. The differences in formulation between Apollonius and Euclid regarding equality by superposition are highlighted in Rashed's commentary. The use that Apollonius makes of the concept of superposition implies some idea (even though never explicated) of motion:

> La définition de l'égalité par superposition...peut encore se dire ainsi: deux sections—ou portions—coniques sont dites égales si elles coïncident parfaitement une fois que l'une est amenée sur l'autre par un déplacement, de sort que leurs contours s'identifient. [11]

In this definition:

(1) no concept of magnitude or measure is ever introduced;
(2) an idea of motion in the sense of a transformation is presumed but never explicated by Apollonius;
(3) there are no operating concepts, a fact which thus necessitates the integration of other properties whose use is more directly operative (the symptoms); and
(4) it is necessary to insert other procedures that integrate the concept of motion.

These observations will become clearer if we examine some of the first propositions of book 6.

Propositions 6.1 and 6.2 concern the equality of two conic sections (the parabola in the first and the hyperbola in the second). It is proven, for example, that two parabolas are equal if and only if they have the same *latus rectum*. Recall that the *latus rectum* of a parabola has the following property [see Figure 1]: the *latus rectum* is defined as that segment c, where B is a point of the parabola, C the corresponding point on the axis, and A the vertex of the parabola, such that CB is the mean proportional between AC and c. In modern terms, if we set $BC = x$ and $AC = y$, then we have $x^2 = cy$. This is the 'symptom' of the parabola.
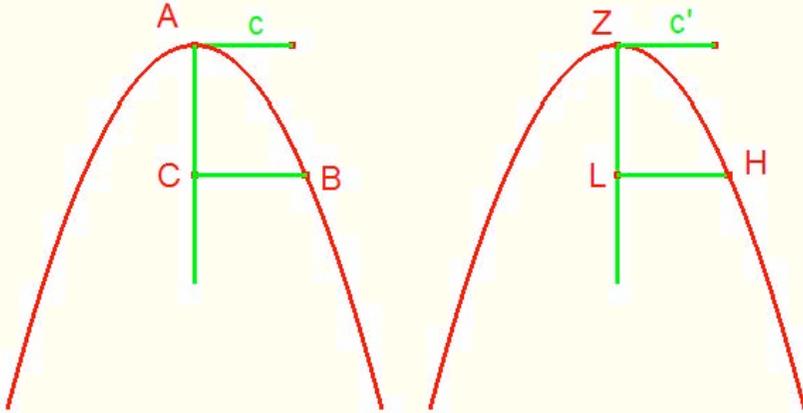
Figure 1

Apollonius' reasoning can be summarized in this way [see Figure 1]. If $c = c'$ and if we transport line AC so that it is superposed on line ZL such that A is carried onto Z, and if we call the point on ZL where C falls L (and thus $ZL = AC$), we will have $CB^2 = cAC = c'ZL = LH^2$. Thus, B too is superposed on H and the two parabolas are pointwise superposed. This reasoning can be clearly inverted.

In analogous fashion, Apollonius proceeds to find the conditions for the equality of two ellipses or two hyperbolas (the central conics), except that in this case what comes into play in addition to the *latus rectum* is either the axis [prop. 6.2] or an arbitrary diameter[corollary to prop. 6.2]: two central conics are equal if and only if their respective 'figures'—that is, the rectangles formed by the axis (or a diameter) and the corresponding *latus rectum*—are equal.

It should be noted that, once these propositions have been proven, Apollonius no longer needs to refer the equality of two conics to the poorly defined concept of 'superposition' but can refer instead directly to their 'symptoms', which in some way correspond to the equations of analytical geometry. Thus, for example, to see that two ellipses are equal it is sufficient to see that their *latera recta* and axes are equal. Rashed rightly notes:

> *La tâche qui est celle d'Apollonius dans le livre VI est donc, pour l'essentiel, de déterminer les conditions pour que les deux sections soient superposables...à l'aide des* symptomata, *sans*

> *toutefois s'intéresser à la nature même de ces transformations ponctuelles.* [6]

In other words, Apollonius, like Euclid, defines equality by means of superposition, which implies an idea, never explicated nor clearly defined, of motion. But he then tries to rid himself of that onerous condition through determining the equality of the conics by means of a simple comparison of magnitudes (segments or surfaces). As Rashed underlines,

> *de fait, au cours des démonstrations, l'égalité/superposition est doublée de l'égalité des aires—ou des longueurs. Apollonius recourt alors aux* symptomata. [11]

This process is completely analogous to that followed by Euclid: thanks to the theorems in the equality of triangles [*Elem.* 1.4 and 1.8], verification of the equality of two triangles (and, thus, of any two polygons) is reduced to the equality of segments and angles. A similar procedure is used in book 3 regarding circles, whose equality is attributed (in this case starting from the definitions) to the equality of the diameters.

It is interesting to note that, while this technique makes it possible for Apollonius to free himself from having to resort to superposition any time that two conics must be compared in their entirety, it loses its efficaciousness when he has to compare portions of conics: in this case, it is necessary to go back to the original definition of equality and thus to superposition. In this sense, the idea of superposition in book 6 of the *Conics* takes on a role and importance that it never assumed in either Euclid or in the other books of the *Conics*. This has been made clearly evident by Rashed [11]:

> *Malgré l'inspiration euclidienne patente, la définition de l'égalité/superposition recouvrira chez Apollonius plusieurs contenus.*

One instance of the role played by equality by superposition in this book can be found in the proof of what today we would call the symmetry of conics with respect to the axes. For example, let us examine prop. 6.4:

> If there is an ellipse and a line passes through its center such that its extremities end at the section [i.e. the line is a diameter] then it cuts the boundary of the section into two

equal parts, and the surface is also bisected. [Toomer1990, 276]

The proof, which in this proposition is limited to the case in which the diameter is the axis, proceeds by *reductio ad absurdum* [see Figure 2]. Given axis AB, it is supposed by way of *reductio ad absurdum* that, after being turned over, the arc of ellipse AΓB does not coincide[2] with arc AEB, and that it is precisely point Γ where AΓB does not superpose itself on arc AEB. If from Γ we drop the perpendicular ΓΔ to the axis and extend it until it meets arc AEB in a point E, we find, by the definition of axis, that ΓΔE and, further, ΔE are perpendicular to AB; thus, after being turned over ΔΓ coincides with ΔE and Γ coincides with E, contrary to the initial hypothesis.[3]

I believe it evident that such considerations of Apollonius' proofs lead us to imagine a superposition achieved by some motion. Yet, in my opinion, it is not completely clear what Apollonius' idea of that motion was; but at the same time, there seems to be no doubt that, as Rashed shows amply, the point of view expressed in book 6 had a profound influence on later Arabic mathematicians. As Rashed puts it, we are dealing with 'proto-transformations ponctuelles, que les mathématiciens ne cesseront d'exhiber et de développer à partir du IXe siècle à Bagdad' [11].

The definition of similarity, however, is quite different. Apollonius wrote:

> And similar are such that, when ordinates are drawn in them
> to fall on the axes, the ratios of the ordinates to the lengths

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[2] Toomer uses the term 'coincide' in his translation, while Rashed uses 'tombe sur' ('fall on'). The two translations are comparable if we take 'coincide' to mean 'coincide after being turned over'. In any case, Rashed's translation provides a much clearer idea of motion than that implied by Toomer's.

[3] It is also worthwhile observing that today we would have preferred an indirect proof rather than one by *reductio ad absurdum*. Such a proof might have proceeded in this way:

> given any point Γ on arc AΓB, we will show that after turning over
> ΓΔ with respect to axis AB a point on arc AEB is obtained.

But such a proof would have required considering an ellipse as being formed of infinite points, something that was far from the way in which geometric figures were conceived by the Greeks. However, considerations of this sort would inevitably take us too far afield.
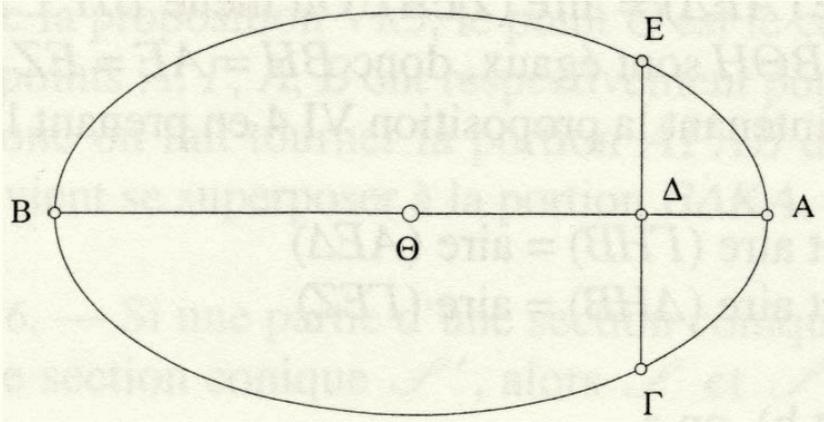
Figure 2

they cut off from the axes from the vertex of the section are
equal to one another, while the ratios to each other of the
portions which the ordinates cut off from the axes are equal
ratios. [Toomer 1990, 264]

In this case, we are dealing with a *functional* definition: to equal ratios
between the abscissas correspond equal ratios between the relative
ordinates. Rashed points out that the concept of similarity between
conic sections is certainly present before Apollonius.[4] Archimedes
stated that all parabolas are similar to each other and, thus, it is
entirely plausible that Apollonius was aware of this fact [Apollonius,
Con. 6.11]. But, as I believe, there is no difficulty in agreeing with
Rashed's statement that

> *rien à notre connaissance ne permet d'affirmer qu'il y a eu
> une étude réglée de la similitude des sections coniques avant
> le livre VI.* [23]

In this case as well, Apollonius moves immediately to substituting
the functional concept of similarity with his verification by means
of the 'symptoms'. Two central conics are similar if and only if
their respective figures—that is, the rectangles formed by the axis
and *latus rectum* [*Con.* 7.12]—are similar. Thus, they are similar
when, given d and d' as the respective axes and the *latera recta c*
and c', $d{:}d' = c{:}c'$. The text continues with several generalizations

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[4] E.g., in his book *On Conoids and Spheroids*: see Heath 1897, 99–150.

(taking into account any given diameters instead of axes) and then it is proved to be impossible for a conic to be similar to a conic of a different name (for example, a parabola can never be similar to a hyperbola, and so forth). Apollonius then deals with segments that are similar or equal in conic sections, first for similar sections and then for dissimilar sections. In this last case (dealt with in propositions 6.23–25), there is a beautiful result: there cannot exist similar segments in dissimilar sections. This result signifies, naturally, that similarity is a local property: if two conics have two similar segments (which are arbitrarily small, we would say), then they are entirely similar. Concluding the part regarding similarity, Apollonius proves that if a right cone is cut with two planes that are parallel to each other, the conics obtained are similar. As Rashed points out, what is in fact proven (using our terminology) is that in this case the two conics are homothetic from the vertex of the cone, with the ratio of homotheity equal to the ratio between the respective distances from the vertex itself.

In contrast, the final part of book 6 presents problems:

- given a conic section and a right cone, cut the cone with a plane so that the intersection is a conic equal to the given one; and
- given a conic section, find a cone similar to a given cone such that the given conic is a section of the cone found.

This kind of problem appears to be meaningful and may in some way provide a clue to Apollonius' aim in writing this book. In a certain sense, it is an inversion of what was done in the first book: while book 1 dealt with constructing the section of a cone as a plane curve, book 6 deals with cutting a given cone according to a given conic section in a plane.

Using the notations shown in Figure 3, where A represents the vertex of the cone, Θ the centre of the circle of the base, and the end points of the diameter of the base B and Γ, the condition under which it is possible to carry out the proposed construction (with d as usual as the diameter and c the *latus rectum*), is that

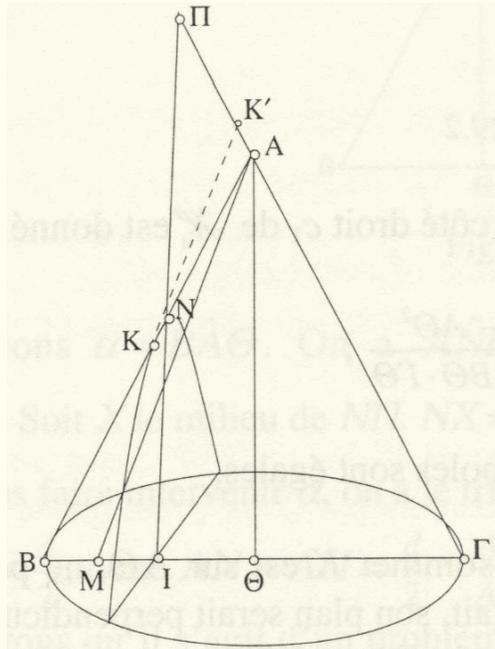$$\frac{d}{c} \geq \frac{\mathrm{A}\Theta^2}{\mathrm{B}\Theta^2}$$

Figure 3

If this condition is satisfied, the construction is done by inserting between the extension of line AΠ and AB a segment (labelled NΠ in Figure 3) parallel to AΘ whose length is equal to d. Rashed makes two observations in this regard that I find particularly interesting.

(1) The first concerns the condition under which the construction can be carried out. The author notes that this condition

> *est équivalent à la condition selon la quelle l'angle entre les asymptotes de l'hyperbole ne doit pas être plus grand que l'angle 2α au sommet du cône. La recherche de la condition de possibilité dans le cas d'un cône oblique aurait été plus difficile: elle ne s'exprime pas en termes d'angle au sommet.* [66]

This observation raises a question which has already been posed by Zeuthen [1886], that is, 'Why is it, having in all preceding books set for himself the more general conditions of oblique cones, that here Apollonius always refers to right cones?' The answer is certainly not that provided by Toomer, who wrote, 'It is easy to see that his

[Apollonius'] solutions in book VI can be extended to the oblique cone' [1990, lviii]. As Rashed notes [66], this is not true at all and the solution becomes quite complicated, at least in this instance, in the passage to the oblique case [see also Brigaglia 1997]. In my opinion, the question remains open; but Rashed in any case provides, with reference to the problems that follow, an interesting hypothesis about this fact.

> (2) As was seen earlier, the construction of the hyperbola requires inserting a segment of a given length that is parallel to one given line. This simple construction is completely absent in Apollonius. Rashed provides a complete proof of this fact, the absence of which appears not to have been noticed by the Arabic translators, although Halley did.

As we said, the final three propositions regard the construction of a right cone (similar to a given one) whose section is a given conic. Here again we can ask why Apollonius limited himself to the case of the right cone.[5] Rashed notes that while in the right case the problem is determinate; in the oblique case, it remains indeterminate. He concludes:

> *C'est précisément par ce caractère d'unicité de la solution que les propositions 31 à 33 diffèrent des propositions 49 et 50 du livre I. C'est ce même caractère qui semble expliquer le choix d'Apollonius du cône droit.*

It would be worthwhile to develop this interesting observation further.

Rashed's presentation of book 6 of the *Conics* ends with a section that is particularly original, 'Le sixième livre et la géométrie proto-transformationnelle'. He writes:

> *Le commentaire systématique du sixième livre révèle en effet qu'il s'agit indubitablement d'une géométrie où l'on procède pragmatiquement par mouvement et transformations ponc-tuelles.* [77]

The word 'pragmatiquement' is especially interesting: in this book, Apollonius makes ample use of concepts such as motion or transformation but without either defining them precisely or using them in a way that is altogether self-conscious. In fact, to find a fully self-

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[5] See 'Remarques sur le propositions 31 à 33' [77].

conscious use of them, we will have to wait for the works of La Hire and then, another two centuries later, of Felix Klein. With regard to the works of La Hire, Rashed writes:

> *Ce regard, même s'il englobe celui d'Apollonius et l'éclaire, n'est cependant pas le sien: ses concepts, ses instruments et son langage sont en effet différents. Cependant, les objets géométriques étudiés dans les Coniques possèdent bien ces propriétés, qui ne seront appréhendées et révélées que par les successeurs d'Apollonius.... C'est donc en restant fidèle à la pensée du mathématicien alexandrin que l'historien peut s'inspirer de ces propriétés, pour mieux pénétrer cette réalité mathématique que celui-ci abordait avec les moyens de la géométrie de son temps. Aussi pour compléter le commentaire du sixième livre, allons-nous le considérer avec d'autres yeux que ceux d'Apollonius, ceux d'un lointain successeur.* [78]

This *lointain successeur* (distant successor) is, in fact, Felix Klein. The lengthy digression [78–83] in which Rashed reconstructs the entire sixth book from the point of view of projection and transformation groups may appear at first to be out of place, but this is not the case. The final lines [83] make the author's motivations clear:

> *Une interprétation de ce type permette de mettre en évidence les transformations ponctuelles sous-jacentes au travail d'Apollonius. Á partir du IXe siècle, ce livre VI, ainsi que les autres travaux d'Apollonius sur les lieux plans, ont incité les géomètres à concevoir les transformations ponctuelles de courbe à courbe (Thābit ibn Qurra et Ibn al-Haytham par exemple).*

To my mind it is precisely here that we find one of the aspects of greatest value in the new translation of the Arabic text of Apollonius, which can be inserted into the imposing context of the Arabic tradition of translating mathematical texts. Interpreted in this light, we can appreciate the work of the great Arabic mathematicians not only as transmitters of Greek thought, but also as original interpreters of the mathematics that was made available to them, interpreters who were capable, through new ideas, of opening new roads—even though a significant portion of them would receive their natural development only much later and in a different culture.

Before going on to a brief look at Book 7, I should like to go back to a central point in Rashed's formulation. Book 6 has

traditionally been considered secondary in the context of Apollonius'
work. As evidence of this, I cite Zeuthen, who says that no real
geometric difficulty is overcome here. However, Apollonius himself
had something to say about this book:

> We have enunciated more than what was composed by others
> among our predecessors....What we have stated on this is
> fuller and clearer than the statements of our predecessors.
> [Toomer 1990, 262]

Fried and Unguru [2001] have a different appreciation as they
say that the importance of book 6 lies in the fact that:

> (1) equality and similarity of conic sections is, for Apollonius, a
>   far more subtle affair than we would like to think;
> (2) the investigation of equality and similarity is necessary to
>   clarify what is meant by a conic section being 'given'; and
> (3) it does not merely elaborate ideas already elaborated in book
>   1 but complements those ideas somewhat in the way Euclid's
>   *Data* complements the *Elements.*

Rashed, however, goes further. Indeed, without pretending to be
completely original, Apollonius does give himself credit for providing
a more complete and systematic organization of the material. This
is precisely what Rashed claims. For him, Apollonius is in search
of new means for extending the study of equality and similarity to
curved figures:

> *Il fallut trouver les moyens de faire correspondre une section
> à une autre, différente, une portion à une autre, différente.*

Thus, Apollonius' aim was

> *trouver les moyens d'étendre aux sections coniques la re-
> cherche accompli pour les figures rectilignes et pour les arcs
> de cercle, et déterminer les conditions requises par une telle
> extension.* [5]

The historical importance of book 6, then, lies in its having paved
the way later taken by numerous Arabic mathematicians:

> *les mathématiciens qui les premiers ont pris davantage de
> distance à l'égard de la géométrie des figures et ont introduit
> mouvement et transformations ponctuelles se sont précisément*

> *référés à ce livre VI—ainsi Thābit ibn Qurra, al-Sijzi, Ibn Hūd, Ibn Abi Jarrāda....* [7]

This is a point of view that I believe is novel, one which only someone like Rashed, who truly knows the contributions of these mathematicians, could provide and which deserves to be examined in greater depth.

The seventh book is quite another story. While the purposes of book 6 are extremely clear, book 7 appears quite difficult to read, not in the sense that it is mathematically difficult but in the sense of trying to understand the aims of its author. In the general introduction to book 1, Apollonius wrote, 'another [*scil.* book 7] [deals] with theorems concerning determinations.'[6] In the accompanying letter from Apollonius to Attalus, he also wrote:

> Peace be with you.... In this book are many wonderful and beautiful things on the topic of the diameters and the figures constructed on them, set out in detail. All of this is of great use in many types of problems, and there is much need for it in the kind of problems which occur in conic sections which we mentioned, among those which will be discussed and proven in the eighth book of this treatise. [Toomer 1990, 382]

Here the word 'diorismes' ('determinations') signifies the determination of a problem's conditions of solvability. Thus, we are dealing with a book in which are determined the range of possible variation for values relative to the diameters and *latera recta* of conic sections.

As Rashed rightly observes, it is very difficult to comprehend fully the significance of the choices made by Apollonius without having access to the eighth book (which, as mentioned, has been lost definitively), because it in fact appears that we are dealing with elements that are very closely tied to the solution of problems given in that eighth book. All of this is clearly highlighted by Rashed:

> *Quant à l'usage qui serait fait de ces théorèmes au huitième livre, nous l'ignorons puisque celui-ci est définitivement perdu et qu'aucun témoignage fiable ne nous est parvenu à son propos.* [241]

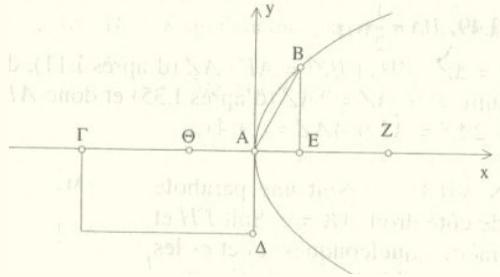[6] 'determinations': *diorismes*, in Rashed's translation.

Figure 4

Thus, book 7 has to be read on its own, since no references to book 8 are possible. This is done in an exemplary way in the rest of the text that follows. The hinge of Rashed's interpretation is that it effectively consists in the study of the variation of several magnitudes tied to diameters and *latera recta* and, thus, to the determination of the maximum and minimum values that these can reach. This provides a point of continuity with book 5, which is dedicated to the determination of maxima and minima of magnitudes such as the distance of a point from the points of a conic section:

> *Le livre VII est dans une certaine continuité avec le livre V. Nous avons en effet montré que, dans ce dernier, Apollonius étudie la variation de la distance d'un point donné aux points d'une section conique. Mais cette continuité s'observe aussi dans la formulation des propositions...et dans la communauté du lexique.* [245]

It seems to me that this continuity is amply proven by Rashed's examination of the text, with perhaps one caveat: while book 5 is self-contained and its purpose lies in the search for maxima and minima of some magnitudes found in it, book 7 is completely oriented towards book 8 and is, therefore, much more difficult for a modern reader who does not have access to that last book, to grasp fully the beauty and depth of the theorems that are contained in it.

It is precisely these characteristics that prevent me from going into technical details. One central point of the second proposition in this book is the introduction of a new magnitude that Rashed translates as 'segment semblable en proportion' [251] (which Toomer,

following Halley, translated as 'homologue'). This is what is at issue
[see Figure 4]:[7]

> Let there be hyperbola H with transverse axis d_0 and *latus
> rectum* c_0, and let Θ be on diameter $A\Gamma$ such that there is

$$\frac{\Theta\Gamma}{\Theta A} = \frac{d_0}{c_0} \ .$$

> We would call ΘA the segment 'similar in proportion'.

Five lemma-like propositions are dedicated to this magnitude. Book 7
then goes on with a group of propositions (from 6.6 to 7.20) dedicated
to the determination of formulas relative to ratios between different
magnitudes like

$$\frac{d_0{}^2}{(d - d')^2} \ .^{[8]}$$

The second part of the book is the more substantial one, and is dedi-
cated to the study of the variation of magnitudes such as diameters,
associated *latera recta*, their sums, differences, products or ratios.

This is the part of book 7 that is most directly connected to book
5, itself dedicated to the determination of the maxima and minima
that are (presumably) necessary for determining the conditions of
solvability for the problems treated in the lost book 8. It is precisely
here that we see the important role played by the attempts to re-
construct the long lost book 8 in order to understand the kind of
interpretation that various readers have given to the theorems set
out in book 7. Famous among such attempts are those of Halley and
the 11th-century Arabic mathematician Ibn al-Haytham. Rashed
refers in particular to the latter in the reconstruction proposed in
this present volume.[9] Rashed cites two examples that I believe it is
interesting to repeat here:

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[7] I will discuss only the case of the hyperbola, the analogous one for the ellipse
is given in 7.3.

[8] Here d_0 represents as before the transverse axis, while d and d' represent
any diameter and its conjugate.

[9] He had earlier proposed and commented on this reconstruction in Rashed
2000.

(1) Given a central conic, find a point such that the diameter d drawn from this point and the associated *latus rectum c* satisfy the equation $dc = k$, with k given.

(2) Given a central conic with transverse axis d_0[10] and associated *latus rectum c_0*, find a diameter d and its associated *latus rectum c* such that $d + c = k$, with k given.

From the use that the Arabic mathematician makes of the propositions of book 7 to solve these and other problems, Rashed draws interesting conclusions:

> *Par « théorèmes relatifs aux diorismes », il semble donc que Apollonius entende deux choses à la fois. Il s'agit de propositions qui d'une part renferment elles-mêmes des diorismes, et qui d'autre parte interviennent dans la conception des diorismes lors de la construction des problèmes au moyen de l'intersection des coniques. Tel est bien le cas pour un bon nombre des propositions du septième livre. Or cette dualité de sens, seulement implicite, ne pouvait qu'intriguer les commentateurs.* [248]

To be sure, Rashed's rich commentary made it possible for me to retrace by following his text a magnificent itinerary through Greek mathematics filtered through Arabic culture.

Before finishing with book 7, I should like to note that, as he did in book 6, here too Rashed concludes an introductory commentary with a section ('Étude analytique de la variation des grandeurs associées à d, d', c, c'') that translates the text of the mathematician from Alexandria into modern language. This not only facilitates its comprehension by a modern reader but also makes evident the thread that ties the different mathematical languages together:

> *Grâce à ce modèle, la vérité des propositions se passe de l'appel constant aux figures, ainsi qu'à l'imagination des constructions auxiliaires. Plus importante encore, ce commentaire fait apparaître des liaisons entre les propositions, invisibles à la pure géométrie, et met en évidence des idées majeures qu'on ne pouvait saisir par la démonstration géométrique—ainsi les idées qui président à l'étude de la variation. Cette fois encore, et comme tous le géants qui jalonnent l'histoire des*

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[10] Here I believe there is a typographical error because what is written is d'_0.

> *mathématiques, Apollonius n'œuvre pas seulement dans le présent, mais dans le futur mathématique, avec les moyens du présent. Situation éminemment féconde et extrêmement subtile, qui exige pour être comprise qui soient multipliés les commentaires.* [348]

This is precisely where the fascination of this edition lies: it unites philological rigor with a panoramic point of view which comprises successive developments of Apollonius' ideas without leading to anachronistic flights of fancy that depict mathematicians of classical antiquity as improbable precursors of modernity, but which highlights the thread of continuity that makes the history of mathematics a description of a fascinating adventure in search of those 'hidden harmonies' ('riposte armonie') of which Federigo Enriques spoke so convincingly.

BIBLIOGRAPHY

Brigaglia, A. 1997. 'La ricostruzione dei libri V e VI delle *Coniche* da parte di F. Maurolico'. *Bollettino di Storia delle Scienze Matematiche* 17:267–307.

Ecchellensis, A. (Ibrahim al-Haqilani). 1661. *Apollonius de Perga. Conicorum libri V, VI, VII, paraphraste Abalphato Asphahanensi, nunc primum editi.* Florence.

Fried, M. and Unguru, S. 2001. *Apollonius of Perga's* Conica*: Text, Context, Subtext.* Leiden.

Halley, E. 1710. *Apollonii Pergaei* Conicorum *libri octo et Sereni Antissensis* De sectione cylindri et coni libri duo. Oxford.

Heath, T. L. 1896. *Apollonius: Treatise on Conic Sections.* Cambridge.

————— 1897. *The Works of Archimedes.* Cambridge.

————— 1956 trans. *The Thirteen Books of the* Elements. Mineloa, NY.

Rashed, R. 2000. *Les mathématiques infinitésimal du IXe au XIe siècle: 3. L'achèvement de l'ouvrage des* Coniques. London.

Toomer, G. T. 1990. *Apollonius:* Conics *Books V to VII.* Heidelberg.

ver Eecke, P. 1923. Les Coniques *d'Apollonius de Pergé.* Bruges. Repr. 1959, Paris.

Zeuthen, H. G. 1886. *Die Lehre von Kegelschnitten im Alterturm.* Copenhagen.