Abū Kāmil. Algèbre et analyse Diophantienne. Édition, traduction et commentaire by Roshdi Rashed[*]

Scientia Graeco-Arabica 9. Berlin/Boston: Walter de Gruyter, 2012. Pp. xvi + 819. 978–3–11–029561–0. Cloth €140.19

Reviewed by
Jeffrey A. Oaks
University of Indianapolis
oaks@uindy.edu

After al-Khwārizmī, Abū Kāmil (late ninth century) is the next Arabic author whose book on algebra is extant in its entirety. Where al-Khwārizmī's book was deliberately brief, Abū Kāmil's *Kitāb al-jabr wa'l-muqābala* (*Book of Algebra)* occupies 111 folios in the only surviving manuscript.[1] That is long enough for Abū Kāmil to show features and techniques omitted by al-Khwārizmī and to exhibit his own originality with regard to proofs, irrational numbers, and the manipulation of algebraic expressions.

Abū Kāmil's influence was deservedly almost as far-reaching as al-Khwārizmī's. The *Book of Algebra* became a major influence on such well-known figures as al-Karajī, al-Samaw'al, and Ibn al-Bannā', as well as on lesser ones like ʿAlī al-Sulamī and Ibn Badr. Most of the book was translated into Latin in the 12th century and whole portions found their way into the *Liber Mahameleth* in the 12th century, into Fibonacci's *Liber abaci* and *De practica geometrie* in the 13th century, and into Jean de Murs' *Liber quadripartitum* in the 14th century. From there its influence spread through Italian abbacus texts to Luca Pacioli's *Summa de arithmetica* of 1494 and into 16th-century European algebra. We also possess several manuscripts of a Hebrew translation made before 1475, possibly in Spain.[2]

---

[*] English translations are mine unless noted otherwise. In referring to Rashed's text, '579.13', e.g., means 'page 579, line 13'.

[1] Istanbul, Beyazit Library, Kara Muṣṭafā Paṣa 379 (now 19046). Copied in AD 1253.

[2] Martin Levey [1966] edited and translated this Hebrew version, which was once thought to have been translated by Mordekhay Finzi.

Two other books of Abū Kāmil are extant. One is his brief *Kitāb al-Ṭayr* (*Book of Birds*), in which he uses algebra to solve problems with several independent unknowns and the other, his *Kitāb al-Misāḥa* (*Book of Mensuration*).[3]

Roshdi Rashed has brought together critical editions, translations, and commentaries of the *Book of Algebra* and the *Book of Birds*. In his introduction [1–31], he provides what information we have on Abū Kāmil's life, his works, and his influence. The commentary occupies chapters 1 through 7 [33–239] and this is followed by editions with facing French translations of both the *Book of Algebra* and the *Book of Birds* [242–761]. Rashed concludes with an extract of Abū Kāmil from al-Samawʾal's 12th-century *al-Bāhir fī ʿilm al-ḥisāb*, some notes, an Arabic-French glossary, indices, and a bibliography [763–819].

The contents of Abū Kāmil's books

The *Book of Algebra* is composed of three parts, each of which can be regarded as a treatise on its own:

(1) Pp. 242–521   The 'algebra proper' is modeled on the first half of al-Khwārizmī's book. The names of the powers are given, then the six equations are classified and their solutions are given with geometric proofs. Many sample calculations with polynomials and roots follow, often with proofs. Then comes a collection of worked-out problems with various proofs scattered throughout. By Rashed's count there are 70 problems in total.[4] Abū Kāmil is unusual in that he gives two or more solutions to many problems.

(2) Pp. 522–577   'On the Pentagon and Decagon'. Here Abū Kāmil solves 20 problems in geometry *via* algebra.

(3) Pp. 578–729   Abū Kāmil solves 43 indeterminate problems by algebra, followed by 27 assorted determinate problems, some of which are not solved by algebra. After this are some problems in numerical progressions. Rashed lumps all these problems together under the chapter 'Analyse indéterminée'. As Jacques Sesiano [1977] has shown, Abū Kāmil was evidently not familiar with Diophantus'

---

[3] Jacques Sesiano [2013] has recently published a critical edition with English translation of the *Kitāb al-Misāḥa*.

[4] I list 74 problems in Oaks and Alkhateeb 2005, 419–420.

> *Arithmetica* when he wrote this book, even if Qusṭā ibn Lūqā had
> translated the *Arithmetica* two or three decades earlier.[5]

The *Book of Birds* [732–761] consists of six problems solved by algebra
with multiple independent unknowns. In the first problem, 100 birds of
three species are purchased for 100 *dirhams*. Ducks are 5 *dirhams* each,
20 sparrows cost a *dirham*, and chickens are a *dirham* each. How many
of each species are bought? In the solution, the number of ducks is named
'a thing' and the number of sparrows 'a dīnār', and the problem is solved
with these two unknowns.

Rashed's editions and translations

We are told [ix: cf. 27, 30] that

> *[l]e lecteur trouvera ici l'*editio princeps *du livre d'algèbre d'Abū Kāmil et de
> son autre livre,* Sur les volatiles*, ainsi que leur traduction intégrale.*

> [t]he reader will find here the *editio princeps* of the *Book of Algebra* by Abū
> Kāmil and of his other book, *On Birds*, as well as their complete translation.

It is true that no complete edition of the *Algebra* has been published before
but Rashed should have mentioned Sami Chalhoub's edition and German
translation of part 1 [2004] and especially the facsimile of the entire book
published by Jan Hogendijk [1986]. Historians have thus had easy access to
the whole text of the *Algebra* for more than a quarter century.

In comparing several pages of Rashed's edition of the *Algebra* with the
facsimile, I found only two minor errors: he does not indicate that the «min»
which he added at page 475.9 is not in the manuscript and that he has
corrected «sitta» ('six') to «arbaʿa» ('four') at page 515.9, though «sitta» is
correct. Overall the edition is excellent.

The French translation fills a real need. The previous English translations of
parts of the *Algebra* are less than adequate and Rashed's version is clear and
literal. My only quibble would be about the translations of certain terms; but
this is a consequence of his interpretation of algebra, which I discuss below.

---

[5] Rashed does not cite this article.

The commentary

Rashed devotes over 200 pages to explain the mathematics in Abū Kāmil's *Algebra*. Throughout the commentary, he represents Abū Kāmil's calculations with modern symbols, which should give us a quick guide to Abū Kāmil's 'rhetorical' mathematics, that is, mathematics written without notation. But despite his warning that these symbols carry with them concepts alien to medieval mathematics—see, e.g.:

> *Le symbolisme autorise en effet des généralisations, des itérations, des déductions, etc., que la langue naturelle est souvent inapte à opérer. Et, qui plus est, ce modèle interprétatif élaboré à partir de l'algèbre symbolique s'avère parfois inefficace.* [ix–x]

—Rashed consistently interprets Abū Kāmil's words through them. This distorts the text in two ways. First, Rashed applies his modern symbols to both medieval algebra and medieval arithmetic, which both obscures the structure of algebraic problem solving and levels the distinction between medieval and modern algebra. Second, this symbolic algebra serves as the foundation upon which he interprets the indeterminate problems in part 3 using terms of modern algebraic geometry.

Another fundamental problem of interpretation is Rashed's view that Arabic algebra is a scientific 'theory of equations' centered on the classification, solutions, and proofs of the six canonical equations. In fact, Arabic algebra was fundamentally a numerical problem solving technique and the solutions to the six equations are among the rules necessary for its implementation. This mis-orientation together with his notational transgressions form the basis for Rashed's claim that Abū Kāmil founded indeterminate analysis.

These matters of interpretation are not easily explained in a short review, so I will spell out the causes and consequences of Rashed's misconceptions in some detail. I will begin by describing the basic structure of medieval algebraic solutions to problems.

Modern algebra and medieval mathematics

Medieval Arabic algebra was part of arithmetic. As a technique for solving numerical problems, it was practiced alongside methods such as single and double false position, working backwards, and 'analysis'. In these methods, one calculates directly with the numbers given in a problem to get the

answer. What distinguishes a solution by algebra (*al-jabr wa'l-muqābala* or sometimes just *al-jabr*) is that an unknown number is *named* and an *equation* is set up and then solved.

The solution to a problem in medieval algebra can be divided into three stages:

Stage 1 An unknown number is named in terms of the powers *jidhr/ shay'* (root/thing, akin to our *x*), *māl* (plural *amwāl*; a sum of money, x^2) , *ka'b* (cube, x^3), and so on. Then operations are performed to set up an equation that is expressed in terms of these names. This is ideally a polynomial equation.

Stage 2 The equation is simplified, using *al-jabr* (restoration) and *al-muqābala* (confrontation), to one of the six types listed by al-Khwārizmī, Abū Kāmil and others:

Simple equations
> *amwāl* equal roots ($ax^2 = bx$)
> *amwāl* equal number ($ax^2 = b$)
> roots equal number ($ax = b$)

Composite equations
> *amwāl* and roots equal number ($ax^2 + bx = c$)
> *amwāl* and number equal roots ($ax^2 + b = cx$)
> roots and number equal *amwāl* ($ax + b = cx^2$)

Stage 3 The simplified equation is solved following the numerical procedure given in the beginning of the book. Some books give geometric proofs that these rules work.

These stages were followed in the solutions to problems in all books in Arabic algebra and in book chapters devoted to algebra beginning with al-Khwārizmī and Abū Kāmil.[6] Also, many books in Arabic arithmetic show solutions by different methods to the same enunciation and these stages are followed in solutions worked out 'by algebra'.[7]

---

[6] The surviving books on algebra by Ibn Turk and Thabit ibn Qurra do not contain worked-out problems. They show only the solutions and proofs for the rules to solve simplified equations for stage 3.

[7] Al-Ḥaṣṣar (late 12th century), Ibn al-Yāsamīn (d. 1204), al-Fārisī (d. *ca* 1320), and al-Kāshī (d. 1429) are some authors who show solutions by multiple methods.

As an example, here is the enunciation and first solution to Abū Kāmil's problem <7>:[8]

> [Enunciation]
>> If [someone] said to you, 'ten', you divided it into two parts. You multiplied each part by itself and you cast away the smaller from the larger, leaving eighty.
>
> [Stage 1]
>> Its rule is that you make the smaller part a thing [x] and the larger ten less a thing [$10 - x$]. So you multiply ten less a thing by itself to get a hundred *dirhams* and a *māl* less twenty things [$100 + x^2 - 20x$]. Then you multiply a thing by itself to get a *māl*. Subtract it from a hundred *dirhams* and a *māl* less twenty things, leaving hundred *dirhams* less twenty things equal eighty *dirhams* [$100 - 20x = 80$].
>
> [Stage 2]
>> So restore the hundred *dirhams* by the twenty things and add it to the eighty to get twenty things and eighty *dirhams* equal a hundred *dirhams* [$20x + 80 = 100$]. Cast away eighty from a hundred, leaving twenty *dirhams* equal twenty things [$20 = 20x$].
>
> [Stage 3]
>> So the thing is one, which is the smaller part, and the larger part is nine, which is the remainder from the ten. [335–337]

It is important to observe that the enunciation is a question in arithmetic that contains no algebraic terms. It asks for the two unnamed parts of 10 that satisfy a particular condition. Algebra—and by 'algebra' I mean the specific technique called in Arabic *al-jabr wa'l-muqābala* and not a modern, more inclusive notion of 'algebra'—only makes its appearance in the beginning of the solution. There one of the parts is named a 'thing', making the other 'ten less a thing'. The equation ($100 - 20x = 80$) is set up at the end of stage 1 *after* working through the operations and it simplifies to $20 = 20x$ in stage 2. This is one of the three simple types, so stage 3 is trivial. There is no need to follow a procedure to 'halve the roots', and so on.

In his commentary Rashed expresses the enunciation to problem <7> as the modern system of equations:

---

[8] Rashed numbers the problems using angle brackets.

$$\begin{cases} 10 = x + y \\ y^2 - x^2 = 80. \end{cases}$$

Just by writing this down he has already named the parts x and y and he has transformed the operations of the enunciation into equations. He is thus already about half way through a modern algebraic solution to the problem. He continues with 'Alors $100 - 20x = 80$, d'où $x = 1$' [57]. By skipping over the naming of the unknown and the subsequent operations in stage 1, he is able to bridge his own modern equations with the modern version of the one that Abū Kāmil actually sets up.

There are alternate ways to name the unknowns. Abū Kāmil in fact gives two more solutions to this problem with different namings. In the second solution, he switches the assignments:

> So if we made the larger part a thing $[x]$ and the smaller ten less a thing $[10 - x]$...

Here he has set up the equation 'twenty things less a hundred *dirhams* equal eighty *dirhams*' ($20x - 100 = 80$) and then x is found to be 9. In the third solution, he names the parts according to the habit of the 'arithmeticians' (*ḥussāb*):

> You make one of the parts five and a thing $[5 + x]$ and the other five less a thing $[5 - x]$...

This time the equation is set up as 'twenty things equal eighty *dirhams*' ($20x = 80$), so x is 4, making the parts again 1 and 9. This last way of naming the parts is not compatible with Rashed's system of equations, since the 'thing' does not correspond to his x or y. For this reason, he regards it as a change of variables: $x = 5 - u$, $y = 5 + u$ [58].

In many problems, Rashed is explicit in speaking of 'equations' in the enunciation and of a 'change of variables' that in reality is the naming of an unknown. For example, he begins his commentary to problem <67> with:

> On considère le système d'équations
> $$\begin{cases} u + v = 10 \\ u + 2\sqrt{u} = v - 2\sqrt{v}. \end{cases}$$
> Abū Kāmil pose $u = 5 - x$ et $v = 5 + x$. [104]

On the next page he writes [105]:

> *Pour résoudre le système initial sans changement de variable à la Diophante....*
>
> To solve the initial system without a change of variable as in Diophantus....

(We will see below what Diophantus has to do with this.)

I should give one more example, this time to show how Rashed misinterprets the operations in stage 1 that lead to the equation. The enunciation to problem <8> is:

> And if [someone] said to you, 'ten', you divided it into two parts. You divided each one of them by the other, so they resulted in four and a fourth. [337]

Rashed's symbolic version, which again entails naming the parts x and y and setting up equations, is helpful in seeing what is being asked:

$$\begin{cases} 10 = x + y \\ \frac{x}{y} + \frac{y}{x} = 4 + \frac{1}{4}. \end{cases} \text{[58]}$$

In the first solution Abū Kāmil names the parts 'a thing' (x) and 'ten less a thing' ($10-x$). To make one side of the equation, he squares both x and $10-x$ and adds them to get $100 + 2x^2 - 20x$. Then, for the other side, he multiplies the two parts to get $10x - x^2$, which is then multiplied by the $4\frac{1}{4}$. The equation is then set up at the end of stage 1 as $42\frac{1}{2}x - 4\frac{1}{4}x^2 = 100 + 2x^2 - 20x$. This simplifies to $x^2 + 16 = 10x$ in stage 2 and is solved in stage 3 using the prescribed procedure.

Rashed explains the solution in this way:

> *Or* $x^2 + y^2 = xy(\frac{x}{y} + \frac{y}{x})$. *On a donc*
>
> $$x^2 + (10 - x)^2 = (4 + \frac{1}{4})x(10 - x),$$
>
> *d'où*
>
> $$x^2 + 16 = 10x$$
>
> *et* $x = 2$ *ou* 8 *si on utilise l'algorithme.* [58]

Rashed's version may follow the same underlying line of reasoning as Abū Kāmil's version but the two are worlds apart in their execution. Abū Kāmil separately constructs the two sides of his equation by performing operations before he finally states it. Rashed, on the other hand, connects his system of

equations with Abū Kāmil's simplified equation $x^2 + 16 = 10x$ by writing an identity followed by an equation, neither of which is stated in the original.

Abū Kāmil solves this particular problem in five different ways. In the second solution, he again names the parts according to the habit of the arithmeticians. This time naming them 'five and a thing' and 'five less a thing' makes the solution easier because the simplified equation has only two terms:

$$56\frac{1}{4} = 6\frac{1}{4}x^2.$$

Following Rashed's numbering of the problems in part 1, there are 34 enunciations that ask for the two parts of 10, with 2 more in part 3. There is one other common problem type, which occurs 27 times in part 1 and 26 times in part 3. These enunciations ask for an unknown *māl*, where «māl» is here a common term meaning 'quantity' or 'sum of money'; it is not the algebraic name of the second degree unknown as Rashed presumes.[9]

One of several reasons that the algebraic reading is untenable [see Oaks and Alkhateeb 2005, Oaks 2010] is that this *māl* is named in terms of the algebraic powers in the beginning of the solution, just like the parts of 10. In part 1, Abū Kāmil names the *māl* 'a thing' (x) 19 times, which Rashed regards as 'un changement de variable délibéré, purement algébrique, $x^2 \rightarrow x$' [332]. In other solutions, it is named 'a *māl* less twenty-four *dirhams*' ($x^2 - 24$), 'two *māls*' ($2x^2$), 'a third of a *māl*' ($\frac{1}{3}x^2$), and 'half a *māl*' ($\frac{1}{2}x^2$). Most telling is the fact that he names the *māl* (quantity) a *māl* (x^2) six times. If the 'māl' in the enunciation were already the algebraic name, there would be no need to rename it as itself in the beginning of the solution. This is even more common in the chapter on indeterminate problems. There Abū Kāmil begins the solutions to 20 problems with 'So you make your *māl* a *māl*.' One of these problems is translated on page 41 below.

Throughout his commentary, Rashed gives no hint that he understands medieval algebraic problem solving. By treating the enunciation to a problem as equivalent to his equations, he misses or misunderstands the naming of

---

[9] This is in contrast to Rashed's edition and translation of al-Khwārizmi's *Algebra* [2007] where he translates «māl» in the enunciation of 13 problems correctly as 'bien' ('amount' in the English translation). The enunciations of the other 12 problems of this type also involve the square root (*jidhr*) of this quantity (*māl*), which causes Rashed to mistake them both for the names of the algebraic powers.

the unknown and he distorts the working out of operations that lead to the setting up of the equation in stage 1. Throughout his book, in fact, Rashed sees just about any kind of numerical equating, including arithmetical operations, as an equation. The specific and deliberate mode of stating equations in medieval algebra, which is characterized by both vocabulary and context [see Oaks 2010], becomes lost in Rashed's sea of symbols.

I will give one other example of Rashed turning arithmetic into algebra. Abū Kāmil gives the solutions to the three composite equations in the beginning of his *Algebra* and each solution is presented as a sequence of numerical operations. His procedure for the sample equation $x^2 + 10x = 39$, for example, unfolds this way: $10 \div 2 = 5$; $5^2 = 25$; $39 + 25 = 64$; $\sqrt{64} = 8$; $8 - 5 = 3$, so $x = 3$ and $x^2 = 9$. Rashed, instead, gives this algebraic reading in his footnote [250n8]: 'On a $(x + 5)^2 = x^2 + 25 + 10x = 39 + 25 = 64$, donc $x + 5 = 8$, d'où $x = 3$ et $x^2 = 9$'. Abū Kāmil does not square the binomial $x + 5$ or replace $x^2 + 10x$ with 39. In fact, the procedure never deals with the algebraic powers at all. Rashed gives the same kinds of invented algebraic versions for other procedures in footnotes 10, 12, 17, and 23 on the next few pages. In footnotes 9, 11, 13, and 18, however, he gives purely arithmetical and, thus, more appropriate explanations for Abū Kāmil's procedures for finding the *māl* (x^2) directly.

The invention of indeterminate analysis?

Rashed's misconceptions not only blind him to the structure of medieval algebraic solutions, they also have serious consequences for his interpretation and assessment of Abū Kāmil's chapter on indeterminate analysis. In his estimation,

> *[l]e troisième livre de l'*Algèbre *d'Abū Kāmil représente un événement mathématique majeur, dont l'importance n'a pas échappé aux successeurs. C'est dans ce livre en effet que l'on rencontre la première étude délibérément et entièrement consacrée à l'analyse indéterminée rationnelle.* [145: cf. vii]

> [t]he third book of Abū Kāmil's *Algebra* represents a major mathematical event, the importance of which did not escape his successors. It is in this book, indeed, that one encounters the first study deliberately and entirely devoted to rational indeterminate analysis.

Here Rashed has found a superficial reason to dismiss Diophantus, who

> *ne distingue pas plus entre problèmes déterminés et problèmes indéterminés qu'entre problèmes possibles et problèmes impossibles.*

> does not distinguish between determinate problems and indeterminate problems, nor between possible problems and impossible problems.

Rashed makes a claim more worthy of rebuttal farther down the page. Abū Kāmil, he tells us,

> *cherche à constituer, pourrait-on dire, une algèbre des problèmes indéterminés, c'est-à-dire à fonder un nouveau chapitre des mathématiques: l'analyse indéterminée.*

> seeks to establish, one might say, an algebra of indeterminate problems, which is to say to found a new chapter in mathematics: indeterminate analysis.

How can Rashed suggest that Abū Kāmil *founded* indeterminate analysis? Not only did Diophantus devote most of his *Arithmetica* to indeterminate problems but Abū Kāmil himself writes in several places that he borrowed his methods from other arithmeticians (*ḥussāb*). Rashed explains that Abū Kāmil's project is

> *d'algébriser les procédés mis en pratique par les arithméticiens, ou encore de transformer, à l'aide de l'algèbre, des procédés somme toute artisanaux en une science mathématique, et donc en un savoir apodictique.*

> to algebraize the procedures practiced by the arithmeticians or to transform, with the aid of algebra, the overall artisanal procedures into a mathematical science and, therefore, into apodeictic knowledge.[10]

On the contrary, Abū Kāmil makes it clear that the arithmeticians *did* practice algebra. He writes:

> We now explain many indeterminate problems that some arithmeticians call 'fluid'. I mean that one can find many solutions with a sufficient analogy (*qiyās*)[11] and by following a clear procedure. Some of them circulated among arithmeti-

---

[10] He is even more explicit in the preface:

> *Ainsi, en appliquant les procédés de l'algèbre aux problèmes indéterminés, Abū Kāmil conçoit, pour la première fois dans l'histoire, l'analyse indéterminée rationnelle, ou l'analyse diophantienne rationnelle comme on la nomme aujourd'hui.* [vii]

[11] Like other algebraists, Abū Kāmil begins the solutions to his problems with 'Its rule/ inference (*qiyās*) is:' By 'inference' (*qiyās*), he may be referring to the way of naming the unknowns.

cians who solved them by means of types (*al-abwāb*) without establishing their cause (i.e., proof).[12]

As Rashed himself acknowledges [146n3], the 'types' spoken about are the six types of equation listed by al-Khwārizmī and his successors. Abū Kāmil continues in the next paragraph:

> *Nous expliquons également une grande partie de ce que les arithméticiens ont défini dans leurs livres et qu'ils ont fait par types, par l'algèbre et par l'inférence…* [579.18]

> And we likewise explain a large part of what the arithmeticians describe in their books, by means of types (*al-abwāb*), by restoration (*al-jabr*),[13] and by analogy (al-qiyās)….

There are other passages, too, that tell us that the arithmeticians used algebra to solve their problems. Recall that in the third solution to problem <7> Abū Kāmil names the parts of 10 according to their practice. He begins this solution with:

> And if you wanted, you divided the ten into two parts by another division, which is how the arithmeticians usually divide the ten. This method makes it easy to distinguish the larger part from the smaller part and you avoid the problem of halving the roots in many problems.[14] This is that you make one of the parts five and a thing and the other five less a thing.[15] [337.10]

---

[12] Rashed translates this as

> *Nous expliquons maintenant beaucoup de problèmes indéterminés que certains arithméticiens appellent fluides, je veux dire par cela qu'on peut déterminer de nombreuses solutions vraies par une inférence convaincante et une méthode claire; certains de ces problèmes circulent parmi les arithméticiens selon des types, sans qu'ils aient établi la cause à partir de laquelle ils procèdent.* [579.13]

[13] The intended meaning of « al-jabr » is unclear. It may refer to the 'restoration' of diminished terms in stage 2 or to 'algebra'.

[14] By 'halving the roots', Abū Kāmil is referring to the solutions to the composite (three-term) equations, which are more complex than the simple (two-term) equations. Recall that naming the parts in this way avoided a composite equation in problem <8>.

[15] He repeats this in the next problem, at 339.18. Rashed translates the first passage incorrectly as 'Et si tu veux, tu peux partager le dix en deux parties par une autre division qui n'est pas en usage chez les arithméticiens…'.

The naming of unknowns in terms of a 'thing', of course, belongs exclusively to algebra. Also in his chapter on indeterminate problems, Abū Kāmil writes after giving an algebraic solution, 'This procedure is known to the arithmeticians' [657.15]. And after another algebraic solution, he has 'This procedure is that which is applied by the arithmeticians' [677.20]. Last, it should be noted that Abū Kāmil calls *himself* an arithmetician at the end of the book: 'Abū Kāmil Shujāʿ ibn Aslam the arithmetician (*al-ḥāsib*) said:…' [729.3]. Even Rashed acknowledges that arithmeticians might practice algebra, since he translates «ḥisāb», the root of both «al-ḥāsib» and «ḥussāb, as 'calcul et algèbre' [1].

These arithmeticians named unknowns, worked with the six equations, and followed procedures that were later copied by Abū Kāmil. Neither they nor Abū Kāmil include proofs in their works on indeterminate problems, so there is nothing to differentiate their methods. Because the arithmeticians solved indeterminate problems by algebra, Abū Kāmil did not 'algebraize the procedures practiced by the arithmeticians'. Thus, his book does not introduce a new chapter in mathematics. Nor does not 'represent a major mathematical event', at least in the sense that Rashed intends.

The 'method of Diophantus'

The discord between Rashed's interpretation and Abū Kāmil's words can only be explained by addressing Rashed's own conception of Arabic algebra. Before I turn to that, I will review just one more argument that he makes, this one linking the Arabic arithmeticians with Diophantus *via* al-Karajī (early 11th century). Rashed writes this about the arithmeticians' naming of the parts of 10 as 'five and a thing' and 'five less a thing':

> *C'est une méthode qu'al-Karajī appelle plus tard «la méthode de Diophante». Et il est vrai que ce dernier l'applique dans le livre I des* Arithmétiques *pour résoudre l'équation trinôme du second degré.* [58]

> This is a method that al-Karajī later called 'the method of Diophantus'. And it is true that the latter applied it in book 1 of the *Arithmetica* to solve the second-degree trinomial equation.

Let us take a look at this 'method of Diophantus'.

In the beginning of his *al-Fakhrī*, al-Karajī solves and gives proofs for the solutions to the six simplified equations. After covering the sample equation $x^2 + 10x = 39$, he writes:

> And if you wanted to find the root of the *māl* according to the method of Diophantus, you search for a number which, if added to a *māl* and ten things, has a root. It is nothing but twenty-five, which added to a *māl* and ten things has a root, which is a thing and five *dirhams*. And you knew that a *māl* and ten things are thirty-nine units,[16] so, if you removed the *māl* and ten things, and you substituted thirty-nine units, they became sixty-four units. So its root is eight and that equals a thing and five *dirhams*. So the thing equals three *dirhams*, which is the root of the *māl*.[17] [Saidan 1986, 154]

The 'method of Diophantus' here refers to finding a number to add to the $x^2 + 10x$ so that it has a root, as part of solving the simplified equation in stage 3. It is unrelated to the naming the parts of 10, 'five and a thing' and 'five less a thing', that was performed in the beginning of stage 1. The appearance of the term 'a thing and five *dirhams*' in al-Karajī, equivalent to the arithmeticians' 'five and a thing', is merely a coincidence. Likewise for the other solution by the 'method of Diophantus' recorded by al-Karajī, in which he solves $x^2 + 21 = 10x$.

The method of the arithmeticians may be unrelated to the method mentioned by al-Karajī but it is equivalent to Diophantus' naming of his parts in problems 1.27–30. Diophantus' problem 1.29 is a version of Abū Kāmil's problem <7>, in which the two parts together are 20 instead of 10 and the difference of their squares is still 80. Diophantus names the difference between the two parts as '2 ἀριθμοί' (like the Arabic 'two things' or our $2x$).[18] This makes the parts $10 + x$ and $10 - x$, much like the arithmeticians' $5 + x$ and $5 - x$. He then works the operations and sets up his equation similarly, as '40 whole numbers, which are equal to 80 units' ($40x = 80$) [Tannery 1893–1895, 1.64.7].

Rashed's statement that al-Karajī's 'method of Diophantus' is applied 'in book 1 of the *Arithmetica* to solve the second-degree trinomial equation' contradicts what he wrote earlier in his edition of al-Khwārizmī's *Algebra*. There he argued that al-Karajī 'provided an algebraic reading of the *Arith-*

---

[16] Like many algebraists, al-Karajī uses 'dirhams' and 'units' interchangeably.

[17] Curiously, this solution matches the algebraic explanation that Rashed gave for Abū Kāmil's arithmetical solution to this equation, quoted above.

[18] The term «ἀριθμος», though typically meaning 'whole number', serves in this context as the name given to the first degree unknown in Diophantus' algebra and may prove to be a rational number.

metica', and that 'At no time does Diophantus complete the square; instead in his work the emphasis is on the type of substitution that he uses.' Diophantus of course does not solve any trinomial equations in book 1. He does solve them in later books but without explaining how he found the solutions. It is likely that he solved them by completing the square, as al-Karajī tells us. So one of Rashed's contradictory statements is misleading and the other is simply wrong.

Even with the similarity between the namings of the unknowns by Diophantus and the Arabic arithmeticians, I would not propose, as Rashed does on page 146, that perhaps the latter had Diophantus' text at their disposal before Qusṭā ibn Lūqā translated it into Arabic sometime in the second half of the ninth century. It is more likely that algebra circulated orally among merchants and other practitioners over the course of several centuries and that Diophantus, al-Khwārizmī, and Abū Kāmil took this technique and wrote books on it, introducing more theoretical elements at the same time.

Rashed's view of Arabic algebra

During the past half century or so, historians writing about Arabic algebra have tended to focus on what is most interesting to them from a theoretical perspective. This is the classification of the six canonical equations of degrees 1 and 2, their solutions, and their geometric proofs. Even the books by historians like Youschkevitch [1976] and Berggren [1986], which cover the problem-solving side of algebra in addition to the 'theoretical' aspects, leave one with the impression that Arabic algebra was largely about the six equations and proofs.

Rashed not only gives this same impression, he openly advocates it. Anyone who has read his previous books and articles on Arabic algebra is familiar with his view that al-Khwārizmī invented algebra as a 'theory of equations'.[19] According to him, the core of this invention is the classification of the six equations, their solutions, and proofs.

> To pass over these ideas…would reduce the book [of al-Khwārizmī] to a collection of algebraic techniques, which historians could rapidly assign to the author's predecessors. [Rashed 2009, 49: cf. 35ff in the book under review]

---

[19] First in Rashed 1983 and then in numerous derivative articles, most recently in Rashed 2012 and in the introduction to Rashed 2007.

By defining the core of Arabic algebra in this way, Rashed, more than other historians, marginalizes practical problem solving. But if we read what the medieval texts say, we find that algebra not only originated in problem solving, problem solving remained its main focus throughout its history. The solutions of the six equations are among the rules needed to solve problems, so the introductory chapter in al-Khwārizmī that appears to be devoted to the 'theory of equations' is really primarily a chapter covering the necessary rules for working out problems 'by algebra'.

This artisanal view of Arabic algebra might be a drastic reorientation for some readers, so I will give some of the evidence for it. To start, proofs were not restricted to 'scientific' mathematics but were a feature of practical mathematics as well. I will cite two examples.

In the tenth century, Abū'l-Wafāʾ described the methods of practical geometers in his *What is Needed by the Artisan for Geometric Construction*. Jens Høyrup [1986, 473n27] explains that their proofs were 'of a cut-and-paste character' and that, because of their requirement of 'a concrete rearrangement of parts', they rejected the proofs of geometers working in the Greek tradition. Another example is an anonymous practical text in Greek from late antiquity that shows a geometric proof to a rule for multiplying sexagesimal numbers [Tannery 1893–1895, 2.7–10].

The proofs of al-Khwārizmī are similar to the Greek and Arabic proofs just described in that they do not appeal to Euclid and they compare equal lines and areas without recourse to ratios. Later, as algebra attracted the interest of mathematicians working in the Greek tradition, we find Thabit ibn Qurra, Abū Kāmil, and others writing proofs in the manner of Euclid and citing the *Elements*.

Rashed does not mention that the name given to algebra, «al-jabr wa'l-muqābala», comes from problem solving. The two words making up the phrase refer to the steps applied in stage 2 to simplify equations. They have nothing to do with the classification of the six equations, their solutions, or their proofs.

Neither al-Khwārizmī nor Abū Kāmil mention any 'theory of equations'. In fact, «muʿādala», the Arabic word for 'equation', first appears in the second half of the ninth century in Qusṭā's translation of Diophantus' *Arithmetica*; and the first Arabic algebraist whom we know to use the word is al-Karajī in

the 11th century. Algebraists before then made few references to equations as mathematical objects. When they did, they called them 'problems' (*masāʾil*) [Oaks 2010].

Al-Khwārizmī himself announces the practical purpose of his book in this famous passage from the introduction to his *Algebra*:

> [The caliph] al-Maʾmūn…has encouraged me to write a brief book on algebraic calculation which encompasses the fine and important parts of its calculations that people constantly require in cases of their inheritance, their legacies, their partition, their law-suits, and their trade, and in all their dealings with one another, such as the surveying of land, the digging of canals, geometry, and other various aspects and kinds are concerned.[20] [Rashed 2007, 95]

Consistent with this stated purpose is that the books on algebra by al-Khwārizmī, Abū Kāmil, al-Karajī (his *al-Fakhrī*), Ibn al-Bannāʾ, and many others devote more space to solved problems than they do the 'theoretical' parts. In the book under review, the Arabic text for part 1 contains 40 pages of 'theory' (rules and proofs) followed by 101 pages of problems. That does not include the geometry problems that make up all of part 2 or the arithmetic problems of part 3.

Further, Medieval mathematicians themselves call algebra 'a way to find unknown numbers'.[21] And in texts from before the end of the 12th century, I have found 9 mathematicians who call it a *ṣināʿa* (art, craft, or technique), while only 2 others, both with a practical orientation, call it an *ʿilm* (science, or body of knowledge).[22]

The six equations, with their solutions and proofs, were a *part* of Arabic algebra. To hold that they were its *defining feature* is to project our modern, theoretical attitude about mathematics onto medieval texts, while ignoring all indications to the contrary. Indeed, Rashed is correct that drawing our

---

[20] My translation, adapted from Gutas 1998, 113.

[21] Including al-Fārābī (10th century), al-Karajī (early 11th century), ʿAlī al-Sulamī (11th or 12th century), and al-Khayyām (*ca*1075).

[22] The following call algebra a *ṣināʿa*: Qusṭā ibn Lūqā (9th century), Abū'l-Wafāʾ, the lexicographer al-Khwārizmī (10th century), Ibn Sinā, al-Karajī, al-Bīrūnī, al-Khayyām (11th century), al-Samawʾal, and Sharaf al-Dīn al-Ṭūsī (12th century). Al-Fārābī (10th century) and ʿAlī al-Sulamī (11th or 12th century) call it an *ʿilm*. The meaning of «*ʿilm*» was both slippery and evolving, so one cannot say much about its meaning in this context. But «*ṣināʿa*» is a word that implies a practice rather than a theory.

attention away from the 'theory of equations' 'would reduce the book [of al-Khwārizmī] to a collection of algebraic techniques, which historians could rapidly assign to the author's predecessors'. These 'algebraic techniques' are those of algebraic problem solving and the predecessors, in Abū Kāmil's case, are Diophantus and the Arabic arithmeticians.

Algebraic geometry

Like the problems in Diophantus' *Arithmetica*, the indeterminate problems in Abū Kāmil's chapter are stated in arithmetical terms and are solved *via* algebra by choosing particular values for given numbers and setting up determinate equations. Rashed instead reads the enunciations themselves as modern indeterminate equations.[23] These equations in turn suggest to him a reading of the solutions in terms of algebraic geometry. To see just how far Abū Kāmil's solutions are from such a reading, consider problem <4>:

> If [someone] said to you, 'a *māl*', it has a root. If you subtracted from it six of its roots, then the outcome has a root. This problem is also indeterminate.
>
> Its rule is that you make your *māl* a *māl*. Cast away from it six of its roots, leaving a *māl* less six roots. Its root is smaller than a thing, so make it a thing less four *dirhams* or less five *dirhams* or less three and a third or less whatever number you like, as long as it is larger than one half of the six things that are diminished from the *māl*.
>
> So we make it a thing less four *dirhams* and we multiply it by itself to get a *māl* and sixteen *dirhams* less eight things equal a *māl* less six things [$x^2 + 16 - 8x = x^2 - 6x$]. Confront[24] this to find that the thing is eight and the *māl* is sixty-four. From this cast away six of its roots, which is forty-eight, leaving sixteen and its root is four. [585]

In the solution, it is required that $x^2 - 6x$ has a root. Abū Kāmil tells us to set its root equal to some $x - a$, where this a is any number greater than 3, which is half of the 6. He chooses 4, then he sets up and solves the equation $x^2 + 16 - 8x = x^2 - 6x$ to get $x = 8$. Rashed notes that this method had already been applied by Diophantus [151].

---

[23] For example, on p. 147 he writes of this chapter 'La première partie, qui représente l'essentiel du livre, comprend quarante-trois équations et systèmes d'équations indéterminées.'

[24] Abū Kāmil uses this word, conjugated from «al-muqābala», to mean 'simplify and solve the equation'. See Oaks and Alkhateeb 2007.

In his commentary Rashed consolidates the first six problems with this single equation: '(1) $x^2 + bx + c = y^2 \quad b, c \in Q$'. Where Abū Kāmil's enunciation asks for a single unknown, Rashed instead writes his equation with *two variables*. On this modern algebraic foundation, he then generates this geometric interpretation:

> *C'est la méthode dite «de la corde», que retrouve Abū Kāmil. L'interprétation géométrique qui justifie cette appellation est la suivante: l'équation (1) définit une conique dans le plan dont la clôture projective possède les points, à l'infini. La droite (*) passant par un de ces points coupe la conique en un autre point rationnel.* [151]

> This is the method called 'of the cord' which is found in Abū Kāmil. The geometric interpretation which justifies this name is the following: equation (1) defines a conic in the plane whose projective closure has points $(\pm 1 : 1 : 0)$ at infinity. The line $[x = t$ and $y = t + u]$ passing through one of these points cuts the conic at another rational point.

One must ask how the observation that Rashed's equation defines 'a conic in the plane whose projective closure has points at infinity' helps us understand Abū Kāmil's solution!

It gets worse a few pages down. Rashed begins his explanation of another group of problems with 'The method of Abū Kāmil admits the following geometric interpretation'[25] and in the middle of it all he has this excursion into 20-century mathematics which is in no need of translation:

> *Les équations (2) définissent une surface S intersection de deux cylindres quadratiques dans l'espace des coordonnées (X, Y, Z, T). On définit une application rationnelle f de S dans C par les formules $x = \frac{X^2}{T}, y = \frac{XY}{T}, z = \frac{XZ}{T}$ définies pour $T \neq 0$. On voit que, pour tout $X \neq 0$ et tout point (x, y, z) de C distinct de $(0, 0, 0), f(X, \frac{Xy}{x}, \frac{Xz}{x}, \frac{X^2}{x}) = (x, y, z)$, donc f est surjective en dehors de l'origine; de plus, si $b = a\alpha^2, f(0, Y, \pm \alpha Y, \frac{Y^2}{\alpha}) = (0, 0, 0)$, pour tout $Y \neq 0$, mais l'origine n'est pas dans l'image de f si $\frac{b}{a}$ n'est pas un carré.* [157]

Rashed defends these modern geometric interpretations in the preface. After writing that a symbolic rendering of Abū Kāmil's calculations 's'avère parfois inefficace' ('is sometimes ineffective'), he writes:

> *Le chaptire sur l'analyse indéterminée rationnelle, par exemple, sera mieux éclairé et expliqué par un modèle conçu à partir de la géométrie algébrique,*

---

[25] 'La méthode d'Abū Kāmil admet l'interprétation géométrique suivante'. [157]

> *qui permettra d'identifier les algorithmes appliqués et de comprendre en pro-*
> *fondeur le sens des conditions auxquelles le mathématicien a soumis les*
> *solutions.* [x]

> The chapter on rational indeterminate analysis, for example, will be better
> clarified and explained by a model developed from algebraic geometry which
> allows us to identify the algorithms that were applied and to understand deeply
> the meaning of the conditions under which the mathematician worked out the
> solutions.

Then later on the same page he writes

> *…comment, avec des modèles empruntés à d'autres mathématiques, inventés*
> *dans d'autres contextes inconnus de l'auteur, restituer les significations que*
> *ce dernier a déposées dans son texte?* [x]

> …how, with models borrowed from other mathematics, invented in other con-
> texts unknown to the author, can one restore the meanings that the latter placed
> in his text?

He answers that among other things, it is with

> *…des modèles mathématiques construits à partir des disciplines que ce texte*
> *a contribué à fonder et, donc, appartenant à des mathématiques postérieures*
> *à celui-ci, modèles aptes à révéler la* mathesis *de l'auteur. Dans le cas qui*
> *nous occupe ici, ces modèles sont l'algèbre et la géométrie algébrique. Mais le*
> *recours à ces modèles n'est qu'instrumental….* [x]

> the mathematical models constructed from disciplines that this text has helped
> to found, and thus which belong to later mathematics, models capable of re-
> vealing the *mathesis* of the author. In the case that concerns us here, these
> models are algebra and algebraic geometry. But the use of these models is only
> instrumental….

Who can take seriously the idea that, because Arabic algebra lies as a his-
torical source for modern algebraic geometry, reading Abū Kāmil in those
terms can reveal 'the *mathesis* of the author'? But we are used to this from
Rashed. He has exhibited a string of publications in which he gives a modern
reading of premodern mathematics,[26] always careful in a preface to give a
brief warning that the modern models are anachronistic. Yet, in practice, he
treats them as if they are equivalent to the original.

---

[26] For two recent examples of this kind of modern reading of premodern mathematics,
this time in the case of Apollonius, see Unguru 2010 and Montelle 2011. Rashed has
also done this with Diophantus, Sharaf al-Dīn al-Ṭūsī, and al-Khwārizmī.

The idea of explaining the problems *via* algebraic geometry did not originate with Rashed. Isabella Bashmakova proposed this reading for the problems in Diophantus' *Arithmetica* in 1966, and her book of 1972 in Russian on Diophantus was translated into German in 1974. Rashed, who does not cite Bashmakova, made his first statement of this interpretation in his own edition of the Arabic translation of Diophantus in 1984 and reiterates it now for Abū Kāmil.

Abū Kāmil's Proofs

This review is already too long, so I will be brief here. Rashed discusses Abū Kāmil's proofs in chapter 7, 'La démonstration aux commencements de l'algèbre' [221–239]. He maintains that al-Khwārizmī and his successors

> *adhéraient aux normes de la démonstration héritées de la tradition euclidienne.* [222: cf. vi]

> adhered to the norms of proof inherited from the Euclidean tradition.[27]

But the only trace of anything Euclidean in al-Khwārizmī's proofs is the presence of letters to label vertices in the diagrams, as was first noted by Høyrup [1986, 475]. As mentioned above, al-Khwārizmī's proofs unfold in an intuitive manner uncharacteristic of Euclid. And while Abū Kāmil was one of the first algebraists to write Euclidean-style proofs and proofs that cite Euclid—they are not the same thing!—these proofs betray an uneasy tension between the practical arithmetical foundation of algebra on the one hand and Euclid's geometry and number theory on the other [Oaks 2011]. None of this is mentioned by Rashed.

The presumption of a link between Euclid and al-Khwārizmī together with the conceptual errors that affect his analysis of problems form the foundation for Rashed's analysis of Abū Kāmil's proofs. The result is a thoroughly distorted narrative that I will not attempt to dissect.

Assessment of Abū Kāmil

Rashed stresses the distinction between what he considers to be the apodeictic, scientific algebra created by al-Khwārizmī and extended by Abū Kāmil on the one hand, and the empirical and artisanal practice of Diophantus and the Arabic arithmeticians on the other [vii–viii, 146–147]. In reality, al-Khwārizmī

---

[27] Rashed [2007, 31 ff] makes his case for Euclid's influence on al-Khwārizmī.

and Abū Kāmil both took steps to introduce Greek elements to an algebra of practitioners. Consequently, Rashed's assessment of Abū Kāmil's contributions will be clouded at best. So, instead of reviewing what he says, I will list several of the innovations in Abū Kāmil's books that are not found in earlier works. Some of these were Abū Kāmil's own ideas while others were already in practice before him. I thus break them into two lists.

<div align="center">

INNOVATIONS THAT WERE LIKELY ALREADY A
PART OF ALGEBRA BEFORE ABŪ KĀMIL

</div>

(1) The use of irrational numbers in algebra dates back at least to al-Khwārizmī, if we are to judge by his rules for operating on roots. But Abū Kāmil works with them in intricate ways that are rare even for later authors.

(2) Where al-Khwārizmī worked with only the first two powers of the unknown, Abū Kāmil works with powers up to the eighth degree in the solutions to his problems. Powers up to the sixth had already appeared a couple decades earlier in Qusṭā's translation of Diophantus but Abū Kāmil most likely had not read that book. Because Abū Kāmil works with the higher powers only in his problems, they might have been a part of the native Arabic tradition of algebra before his time.

(3) In Abū Kāmil's problems <39> to <43> in part 3 and in his *Book of Birds*, he gives us the earliest extant use of independent unknowns in Arabic algebra.

<div align="center">

INNOVATIONS THAT APPEAR TO BE
ABŪ KĀMIL'S OWN

</div>

(1) In solving problems, the required unknown is sometimes the *māl* and not the 'thing', so Abū Kāmil gives rules and proofs for finding the *māl* directly for the three composite equations.

(2) Abū Kāmil is clearly the author of most of the 50 proofs in part 1 of the *Algebra*. He gives two main kinds of proof: one that uses a geometric diagram and cites propositions from the *Elements* and another in the style of Euclid's books on number theory. One innovation, not taken up by later algebraists, is his use of algebra to prove propositions in arithmetic.[28]

---

[28] These are described in Oaks 2011. I missed one proof using algebra: see 507.13.

(3) Abū Kāmil goes beyond other algebraists both before and after him by making clever assignments that simplify his algebraic solutions. The arithmeticians had already named the parts of 10 'five and a thing' and 'five less a thing' but Abū Kāmil took this notion further with even more creative namings.

(4) In the solutions to some problems in part 1, Abū Kāmil performs clever manipulations of operations on algebraic expressions in the process of setting up polynomial equations [see Oaks 2009, 198–202].

(5) Jan Hogendijk describes part 2 of the book, which is on the pentagon and decagon as follows:

> Abū Kāmil shows that his algebraic methods can be used to find easy solutions to geometric problems that were difficult or even insoluble for his predecessors. [Hogendijk 1985, first page of introduction]

Typographical mistakes

The algebraic notation in the commentary shows many errors. I did not systematically check all the formulae but the following mistakes surfaced on a quick reading. The last equation on page 55 should have a '$(x - \sqrt{y})$' just before the second equal sign. The last formula for problem <39> on page 81 should be '$y = \sqrt{6} + \sqrt{26}$'. The '$\sqrt{2x}$' in the penultimate equation on page 83 should be '$\sqrt{2}x$'. At the bottom of page 87 and at the top of page 88 the '$\sqrt{12}$' should be '$\sqrt{1/2}$' On page 90 the last equation should be '$x = -1/2 - \sqrt{1/8} + \sqrt{3/8 + \sqrt{20}}$'. The '$-\sqrt{6}$' in the penultimate equation on page 92 should be removed. The last equation on page 235 should end with '$a/x + a/y + 2 = b + 2$'.

Conclusion

Rashed has always worked apart from the larger community of historians of Arabic mathematics and in this book he continues to ignore current scholarship. He again repeats his outdated idea that algebra was created as a science by al-Khwārizmī, using the same turns of phrase that we have been reading for 30 years. Add to this his inability to see the differences between medieval and modern mathematics and it comes as no surprise that his commentary is full of misrepresentations and misinterpretations, especially in parts 1 and 3. The commentary to other parts are not as misleading because there is no

possibility of confusing the enunciations of those problems with equations. In contrast to the commentary, the editions and translations of Abū Kāmil's books are very good.

BIBLIOGRAPHY

Bašmakova, I. G. 1966. 'Diophante et Fermat'. *Revue d'Histoire des Sciences* 19:289–306.

―― 1974. *Diophant und diophantische Gleichungen.* Berlin.

―― 1997. *Diophantus and Diophantine Equations.* Washington.

Berggren, J. L. 1986. *Episodes in the Mathematics of Medieval Islam.* New York/Berlin.

Chalhoub, S. 2004. ed. and trans. *Die Algebra. Kitab al-Gabr wal-muqabala des Abu Kamil Soga ibn Aslam.* Aleppo.

Gutas, D. 1998. *Greek Thought, Arabic Culture: The Graeco-Arabic Translation Movement in Baghdad and Early ʿAbbāsid Society (2nd–4th/ 8th–10th centuries).* London.

Hogendijk, J. P. 1986. ed. *Kitāb fī al-Jabr wa'l-Muqābala. A Facsimile Edition of MS Istanbul, Kara Mustafa Paşa 379, Copied in 1253 C.E..* Frankfurt am Main.

Høyrup, J. 1986. 'Al-Khwârizmî, Ibn Turk, and the *Liber Mensurationum*: On the Origins of Islamic Algebra'. *Erdem* 5:445–484.

Levey, M. 1966. ed. *The Algebra of Abū Kāmil:* Kitāb fī al-jābr wa'l-muqābala *in a Commentary by Mordecai Finzi. Hebrew Text, Translation and Commentary with Special Reference to the Arabic Text.* Madison.

Montelle, C. 2011. rev. Rashed and Bellosta 2010. *Aestimatio* 8:183–187. Available online at http://www.ircps.org/aestimatio/8/183-187.

Oaks, J. A. 2009. 'Polynomials and Equations in Arabic Algebra'. *Archive for History of Exact Sciences* 63:169–203.

―― 2010. 'Equations and Equating in Arabic Mathematics'. *Archives Internationales d'Histoire des Sciences* 60:13–45.

_____ 2011. 'Geometry and Proof in Abū Kāmil's Algebra'. Pp. 234–256 in *Actes du 10ème Colloque Maghrébin sur l'Histoire des Mathématiques Arabes (Tunis, 29–30–31 mai 2010)*. Tunis. Available online at http://pages.uindy.edu/ oaks/Articles/AbuKamilOaks.pdf.

Oaks, J. A. and Haitham M. A. 2005. '*Māl*, Enunciations, and the Prehistory of Arabic Algebra'. *Historia Mathematica* 32:400–425.

_____ 2007. 'Simplifying Equations in Arabic Algebra'. *Historia Mathematica* 34:45–61.

Rashed, R. 1983. 'L'idée de l'algèbre selon al-Khwārizmī.' *Fundamenta Scientiae* 4:87–100.

_____ 1984. *Diophante.* Les arithmétiques. *Texte établi et traduit. Tome 3: Livre 4. Tome 4: Livres 5–7.* Paris.

_____ 2007. *Al-Khwārizmī.* Le commencement de l'algèbre. *Texte établi, traduit et commenté.* Paris.

_____ 2009a. *al-Khwārizmī: The Beginnings of Algebra.* London. An English translation of Rashed 2007 that maintains the same pagination.

_____ 2009b. *Apollonius de Perge.* Coniques. *Tome 2.2: Livre IV. Commentaire historique et mathématique, édition et traduction du texte.* Berlin.

_____ 2012. 'Founding Acts and Major Turning-Points in Arab Mathematics'. Pp. 253–271 in J. Z. Buchwald ed. *A Master of Science History.* Berlin/ New York.

Rashed, R. and Bellosta, H. 2010 *Apollonius de Perge.* La section des droits selon des rapports. Berlin/New York.

Saidan, A. S. 1986. *Tārīkh ʿilm al-jabr fī l-ʿālam al-ʿArabī* (*History of Algebra in Medieval Islam*). 2 vols. Kuwait.

Sesiano, J. 1977. 'Les méthodes d'analyse indéterminée chez Abū Kāmil'. *Centaurus* 21:89–105.

_____ 2013. 'Abū Kāmil's Book on Mensuration'. Pp. 359–408 in N. Sidoli and G. Van Brummelen edd. *From Alexandria, through Baghdad: Surveys and Studies in the Ancient Greek and Medieval Islamic Mathematical Sciences in Honor of J. L. Berggren.* Heidelberg.

Tannery, P. 1893–1895. edd. and trans. *Diophanti Alexandrini opera omnia cum Graeciis commentariis*. 2 vols. Leipzig. Repr. Stuttgart, 1974.

Unguru, S. 2010. rev. Rashed 2009. *Aestimatio* 7:29–42. Available online at http://www.ircps.org/aestimatio/7/29-42.

Youschkevitch, A. P. 1976. *Les Mathématiques Arabes (VIIe–XVe siècles)*. Paris. Translated from the German edition of 1964.