Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry by Glen Van Brummelen

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Spherical trigonometry, once regularly taught in high school, disappeared from the curriculum in the decades following World War II. Yet the applications of this 'forgotten art' are still important and the elegance of the mathematics remains alluring. In *Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry*, Glen Van Brummelen examines the historical background and development of spherical trigonometry through an exploration of the mathematical intricacies. The result is an engaging read that will appeal to historians of science, mathematicians, trigonometry teachers, and anyone interested in the history of mathematics.

Part of the appeal of Van Brummelen's book is its immediate discussion of the practical applications of spherical trigonometry. Van Brummelen handily weaves together mathematical theory with practical needs and offers a picture of the development of trigonometry, leaving no stone unturned in the process. In chapter 1, the author walks through Abū al-Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī's calculation of the distance of the Moon from the Earth. Using a few measurements along with trigonometric functions, Van Brummelen quickly determines the radius of the Earth, which is the first step in determining the distance of the Moon. Modern calculations would involve the use of a sine function on a calculator but Van Brummelen aims to explore mathematics without taking anything on faith [2]. This detailed approach requires Van Brummelen to explain Hipparchus' table of chords, the first known trigonometric table, found in Ptolemy's *Almagest.* Armed with a trigonometric table, Van Brummelen is able to walk the reader through the remainder of al-Bīrūnī's calculation. While this calculation is the aim of the chapter, much is learned along the way about the history of spherical

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trigonometry and Van Brummelen's focus on a specific problem, buttressed by the historical development, is beneficial.

The desire of ancient mathematicians to solve certain problems fueled the exploration of the spherical surface and the development of spherical trigonometry. In chapter 2, Van Brummelen provides an introduction to geocentric astronomy. He explains how Hipparchus calculated the eccentricity of the Sun's orbit using the chord function and how this may have been the first trigonometric problem [29]. With an understanding of celestial motion from a geocentric frame of reference, Van Brummelen turns to spherical geometry and walks through calculations such as that of the smallest and largest possible sums of a triangle on a sphere.

Specific ancient and medieval approaches to spherical trigonometry are discussed in chapters 3 and 4. The author explains Menelaus' Theorem, the primary theorem used in ancient Greece to relate the arcs of great circles on a sphere. In chapter 4, we learn about the Rule of Four Quantities, which replaced Menelaus' Theorem owing to its efficiency of use in astronomical contexts. The Rule of Four Quantities is closely related to the Law of Sines and, as Van Brummelen explains,

> one would expect the Law of Sines, with its simplicity and complete generality, to have transformed medieval astronomy even more than did the Rule of Four Quantities. But science is not always predictable. [64]

The Rule of Four Quantities offered astronomers an economical way to solve for arcs and distances; the Law of Sines did not dislodge this pragmatic method, which was already in place. The sometimes surprising ways in which people have employed some mathematical theories over others to solve problems is one of the themes that Van Brummelen's book successfully explores.

One persistent, and helpful, theme of Van Brummelen's book concerns how practical needs played a role in the development of mathematical theorems. In the medieval Islamic world, for example, spherical trigonometry was instrumental in determining the direction of Mecca (the *qibla*). Al-Bīrūnī deployed four different methods to make this calculation, one of which Van Brummelen works through in detail [66–67]. The need to know in which direction to pray led Islamic mathematicians to produce tables that would point the believer toward Mecca from almost any location on Earth. This was a complex task for any single location; but its importance is demonstrated by the set of tables composed by Shams al-Dīn al-Khalīlī, which contained over 3000 entries [70–71].

Chapters 5 and 6 transition from ancient and medieval spherical trigonometry to the modern approach that is regularly taught in high schools today: examining the triangle on its own and using the six trigonometric functions [74]. Although the development of sine, cosine, and tangent were first seen in Indian astronomy, and although ancient and medieval astronomers used many formulas that are related to the six functions, Van Brummelen shows that it was the Scotsman John Napier (1550–1617) who first systematized these functions. While there remains much more to say about spherical trigonometry, Van Brummelen's discussion of its historical development concludes with Napier's analogies and the work of Jean-Baptiste Joseph Delambre in the 19th century.

The final three chapters focus on special topics: polyhedra, stereographic projection, and stellar navigation. While the historical narrative is set aside in these last three chapters, the book's overall organization is effective. In chapter 7, Van Brummelen explores mathematical theorems that did not have practical applications, starting with finding the area of a spherical polygon and then turning to Euclid's proof of the five regular polyhedra and Euler's polyhedra formula. Chapter 8 examines stereographic projection and the development and use of the astrolabe. The last chapter examines maritime navigation. Here Van Brummelen explains how Venetian merchant ships in the 14th century used methods of navigation that were based on plane trigonometric tables. While it is not always clear how sailors ended up with certain mathematical tables, it is apparent how useful these tables were.

At times, the proofs and mathematical details overshadow the narrative; however, as Van Brummelen states in the preface, this is not a scholarly book. A reader looking for footnotes and a more complete story might be interested in Van Brummelen's earlier book, *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry* [2009]. That said, the real value of this book is its focus on the practical application and its hands-on approach to the mathematics. Various proofs are included throughout every chapter. Since the author recognizes that not all readers will have the time or desire to work through the mathematical details, he includes clear symbols alerting the reader that some of material can be skipped without losing the narrative. This allows the reader to make choices about which proofs to work though with more care. Each chapter ends with a set of exercises that permit the reader to apply the material learned in the chapter. Most of these problems are challenging and in order to make progress the reader must apply the newly learned material in new and thoughtful ways. Van Brummelen states in the preface:

> The experience of wrestling with mathematics (provided that it meets with at least occasional success) can be one of the world's greatest pleasures. [xi]

He does not disappoint: what makes this book unique is the way that the history is coupled together with mathematical problems for the reader to solve. Working through the problems, both theoretical and practical, is an enjoyable task that leads to a deeper understanding and appreciation of the history and development of spherical trigonometry.

BIBLIOGRAPHY

Van Brummelen, G. 2009. *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry.* Princeton.