*Diofanto,* De polygonis numeris. *Introduzione, testo critico, traduzione italiana e commento* by Fabio Acerbi

Pisa/Rome: Fabrizio Serra Editore, 2011. Mathematica Graeca 1. Pp. 247. ISBN 978–88–6227–412–8. Paper €74.00

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(Translated from the Italian by Kim Williams)

In the last 20 years or so, the interest of historians of mathematics in Diophantus has grown in a significant way, changing deeply our perception of his mathematical work. Fabio Acerbi's book contributes to a better knowledge of this Greek mathematician and his methods.

'We know nothing about the life of Diophantus'. These are the opening words of Acerbi's introduction to his commented edition of Diophantus' *De polygonis numeris*. In effect, it is rare for a mathematician of such fame and calibre to be so unknown from a biographical point of view. As to when he lived, the references only allow us to put his *terminus post quem* in the second century BC (thanks to a citation by the mathematician Hypsicles) and his *terminus ante quem* in the second half of the second century AD (thanks to a citation in Ptolemy's *Almagest*), an enormous span of three centuries! Acerbi does not seem very convinced by the attempts at providing more precise dates found in Paul Tannery's important edition of the *Opera omnia* of Diophantus [1893–1895] and I share his doubts. So it seems appropriate to forego any speculative hypothesis and concentrate instead on Diophantus' work.

There are a great number of questions that make this kind of study fascinating. First and foremost, there is the history of the transmission of the text and the events surrounding the circulation of the various manuscript copies. Then, there is the philological reading of the text itself, the distinction between the original text and various interpolations, the comparison of different manuscript copies, and so forth. There is also the reading that is more properly mathematical, in the context of the reality of the period in which the author was working. Finally, there is the reading that the mathematicians


ISSN 1549–4497 (online)
ISSN 1549–4470 (print)

of different periods have given the work and its influence, direct or indirect, on the development of mathematics itself.

Acerbi's long introduction (133 pages out of a total of 243) is actually dedicated in large part to Diophantus' principal work, the *Arithmetica*.[1] He devotes a rich chapter to the transmission of the Greek text of the *Arithmetica* and examines the history of the various manuscript copies that exist in the world.[2] There are 31 such manuscripts (not all of which contain the same range of material) and these are traditionally divided into two streams, Planudean and non-Planudean, according to the whether they descend from the annotated transcription (dated to the end of the 13th century) by the Byzantine intellectual and religious scholar, Maximo Planudes. The manuscript that was studied most by the first modern historian, Paul Tannery is the *Diophantus Matritensis* (Madrid, Biblioteca Nacional, Ms. 4678). It too has an interesting history. Written in the 11th century, it was brought to Messina, probably together with other Greek manuscripts after the Turkish conquest of Constantinople, by Costantino Lascaris, who annotated it. After the failed revolt in Messina, it was carried to Madrid by the Duke of Uzeda and has been in the Royal Library of the Spanish capital since 1712.

The *Arithmetica* has come down to us in mutilated form: of the 13 books that initially comprised it, effectively only six have arrived through Byzantine copies, with four more coming through the Arabic translation of Qusṭā ibn Lūqā, datable to the second half of the 11th century, although the copy studied today was discovered only in 1971.[3] According to the most creditable scholars, the Greek manuscripts transmit books 1–3 and most probably 10–13; the Arabic manuscripts, books 4–7. The *De polygonis numeris*, however, has come down to us in almost all the Greek codices containing the *Arithmetica*.

Acerbi's *De polygonis numeris* is the first Italian edition but interest in it goes beyond the mere fact of language. In effect, each new edition in the Diophantine corpus makes a significant contribution to the solution of the

1 Unfortunately, there is no comparison of the text of the Greek and the Arabic manuscripts, which are only mentioned in passing.

2 For a description of the existing manuscripts, see Allard 1980 (which, however, I was unable to consult). The fascinating story of the transmission of the Greek texts can be found, for example, in Allard 1984, 317–331.

3 See Rashed 1974–1975, 97–122. The transcription is found in Sesiano 1982 and in Rashed 1984.

numerous historiographical problems tied to the work of this 'mysterious' mathematician from Alexandria, problems to which the contribution of a scholar of Acerbi's calibre is significant. Besides the problems of a strictly philological nature (obviously also important for any mathematical interpretation of the work), the works of Diophantus have always aroused passionate debates among historians. Not only is there the difficulty of dating the works with certainty but their place in the context of Greek mathematics also turns out to be quite complex. One of the problems that has always fascinated historians of mathematics concerns the 'algebraic' content of the work. This problem can be viewed from three standpoints:

(a) from that of Greek 'geometric algebra',
(b) from that of Diophantus' influence on Arabic algebraists, and
(c) from that of the impact of the reading of the first editions of the *Arithmetica* on developments of European algebra in the 16th and 17th centuries.

All three standpoints are developed by Acerbi in §2 of the introduction.

The first, (a), raises a topic that is hotly debated. As is known, in book 2 (and in books 5–6) of the *Elements*, Euclid develops a geometrical treatment of the solution to quadratic equations. This fact has given rise to relentless discussions: indeed, his treatment has to be interpreted either as a geometrical translation of a pre-existing algebraic treatment (which must go back to the Pythagoreans and, according to some, even to the Babylonians) or as a radically different vision of the problems being treated. The connection between this kind of problem and those proposed and solved by Diophantus is evident.

As an example, Acerbi cites proposition 2.5 of the *Elements*:

> If a straight line is cut into equal and unequal segments, then the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equals the square on the half. [Heath 1956, 1.382]

In Figure 1, p. 263below, the segment in question is $AD$, which is cut by $B$ into two equal parts and by $C$ into two unequal parts. The proposition states that the rectangle $AC \cdot CD$ plus the square on $BC$ is equal to the square on $BD$. In algebraic terms, if $AC = x$ and $CD = y$, we have the identity:

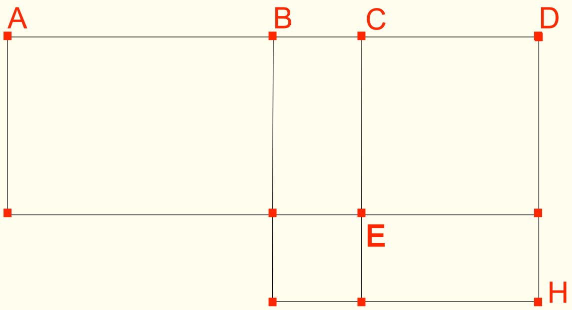



Figure 1. Euclid, *Elem.* 2.5

$$xy + \left(\frac{x-y}{2}\right)^2 = \left(\frac{x+y}{2}\right)^2$$

Diophantus refers to this proposition in the determination of *Arith.* 1.27, in which it is said:

> To find two numbers such that their sum and product are given numbers. The square of half the sum must exceed the product by a square number, ἔστι δὲ τοῦτο πλασματικόν. [Heath 1910, 140]

Problem 1.27 is obviously related to the solution of quadratic equations. If the numbers sought $(x, y)$ have a given sum $(a)$ and a given product $(b)$, then they both satisfy the equation $t^2 - at + b = 0$ and, recalling that, if $x$ and $y$ are the two solutions to the equation, $a$ is their sum and $b$ is their product, the given condition is equivalent to requiring that the discriminant $(a/2)^2 - b$ be a perfect square, thus allowing the problem to have rational solutions (the only kind sought by Diophantus).

It is, for example, precisely in commenting on this proposition (which in his text is number 30) that Bachet de Mezirac puts it into a strict relation to the rule for solving quadratic equations (which he expresses as a canon). It is from here that the first move is made in the long (and still ongoing) tradition of interpreting the work of Diophantus in 'protoalgebraic' terms.

Without dwelling too long on these aspects, I should like to repeat that Acerbi declares himself starkly opposed to such interpretations. He expresses himself thus:

> *Va sottolineato che l'‘algebra geometrica' non è mai stata intesa come un male minore interpretativo atto a rendere ragione agli occhi di un lettore moderno di*

> *certe caratteristiche dei lemmi lineari. La pretesa era invece che i 'greci', come i babilonesi prima di loro, ragionassero davvero algebricamente ma avessero più o meno inconsapevolmente provveduto a coprire il nucleo matematico 'vero' con un velo geometrico. É straordinario che un'interpretazione così rozzamente anacronistica, frutto di un abbaglio storiografico che è durato quasi un secolo e che perdura tuttora tra gli interpreti di Diofanto, possa aver ottenuto credito. La connessione con l'algebra sta ovviamente nella testa degli interpreti moderni, e i testi non offrono il minimo appiglio che corrobori questa tesi.* [18]
>
> It should be underlined that 'geometric algebra' has never been understood as a lesser interpretive evil aimed at rendering to the eyes of a modern reader understanding of certain characteristics of linear lemmas. The claim instead was that the 'Greeks', like the Babylonians before them, reasoned in a truly algebraic way but had covered—more or less consciously—the 'true' mathematical nucleus with a geometric veil. It is astonishing that such a grossly anachronistic interpretation, the fruit of a historiographical blunder that lasted for almost a century and that still today persists among the interpreters of Diophantus, could have acquired credibility. The connection with algebra obviously lies in the minds of modern interpreters and the texts do not offer the least evidence that corroborates this thesis.[4]

Later, I will come back to this debate, which seems to me to be extremely interesting historiographically.

In the meanwhile, I will continue this rapid survey of Acerbi's text, which goes on to examine indeterminant problems, that is, problems admitting an infinite number of solutions. Among these by far the most famous is *Arith.* 2.8. The fame of this problem is primarily due to the notes in the margin of the copy of the celebrated edition by Bachet de Mezirac owned by Fermat:

> *Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duas ejusdem nominis fas est dividere: cujus rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.*
>
> On the other hand, it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally, to infinity, any power except a square into two powers of the same exponent. I have discovered a truly marvellous proof of this, which however the margin is not large enough to contain. [Heath 1910, 144–145]

[4] All unprovenanced translations are my own.

As is known, the theorem stated by Fermat was proved by Andrew Wiles in 1994. But it was all the indeterminant problems (called 'Diophantine problems' in number theory) that guaranteed the Greek mathematician's fame.

It is interesting to retrace with Acerbi the Diophantine solution to 2.8. The problem—I rely on Heath's English translation and its symbols—is this:

> To divide a given square number into two squares.
>
> Given a square number 16.
> $x^2$ one of the required squares. Therefore $16 - x^2$ must be equal to a square.
> Take a square of the form $(mx - 4)^2, m$ being any integer and 4 the number which is the square root of 16, e.g. take $(2x - 4)^2$ and equate it to $16 - x^2$.
> Therefore $4x^2 - 16x + 16 = 16 - x^2$
> or $5x^2 = 16x$ and $x = \frac{16}{5}$.
> The required squares are therefore $\frac{256}{25}$, $\frac{144}{25}$. [Heath 1910, 145–146]

It is best said right away that Heath's translation is extremely liberal, while Acerbi's follows the text much more faithfully. It can in any case be noted that Diophantus' solution is really indeterminant. We need only change the coefficient $m$, giving it arbitrary positive integer values, to obtain an infinity of solutions (with the exclusion of the choice $m = 1$ since in that case the solution is null). This is important because, in this solution, as in few others, the purely exemplifying nature of the choice of the initial numbers is (almost) explicit. Hence, while it is true that 'the fact that other linear substitutions $y = tx - 4$ would have yielded a solution to the original problem as well, tends not to be mentioned' [Schappacher 2005, 13], it is also true that in reality Diophantus indicates here with sufficient clarity the possibility of such an arbitrary choice. Here, the translation is fundamental because what counts lies precisely in the details.

In the Greek text, as transcribed by Tannery [see Plate 1, p. 266below], it can be seen immediately, at least I believe so, that Heath's translation of the third paragraph already begins with the presumption that Diophantus is perfectly aware of the general solution, $(mx - 4)^2$, $m$ being any integer. But this is a translation that is only partly faithful to the original. The more literal translations of «πλάσσω...πλευρά» are those by Norbert Schappacher, Ἐπιτετάχθω δὴ τὸν ις̅ διελεῖν εἰς δύο τετραγώνους.

Καὶ τετάχθω ὁ α^ος Δ^Υ α̅, ὁ ἄρα ἕτερος ἔσται Μ̊ ις̅ ⋀ Δ^Υ α̅· δεήσει ἄρα Μ̊ ις̅ ⋀ Δ^Υ α̅ ἴσας εἶναι □^ῳ.

πλάσσω τὸν □^ον ἀπὸ ς^ῶν ὅσων δήποτε ⋀ τοσούτων Μ ὅσων ἐστὶν ἡ τῶν ις̅ Μ̊ πλευρά· ἔστω ς β̅ ⋀ Μ̊ δ̅. αὐτὸς ἄρα ὁ □^ος ἔσται Δ^Υ δ̅ Μ̊ ις̅ ⋀ ς ις̅· ταῦτα ἴσα Μ̊ ις̅ ⋀ Δ^Υ α̅. κοινὴ προσκείσθω ἡ λεῖψις καὶ ἀπὸ ὁμοίων ὅμοια.

Plate 1. Diophantus, *Arith.* 2.8
[Tannery 1893–1895, 90.11–18]

> Let us take the square of some multiple of $x$ minus the number whose square makes 16. [2005, 13]

and Acerbi,

> Formo il quadrato da quanti si voglia numeri meno tante unità quanto è il lato di 16u.

> I form the square of as many numbers as are desired minus as many units as are in the side of 16u.

I am not at all a philologist but I believe that we can understand how the ‘translation’ of the symbols used by the Greek mathematician into modern algebraic symbols (even though necessary and useful for comprehending the text) can lead to serious misunderstandings. Acerbi’s text is always attentive to these problems. In any case, what appears clearly is the perfect awareness on Diophantus’ part of the completely general nature of his solution and of the existence of infinite solutions to the questions posed. It is now interesting to go back with Acerbi to the problem of the presence of elements of algebra (or at least protoalgebra) in the *Arithmetica*.

In my opinion, two questions should be clearly distinguished: that relating to the effective presence of protoalgebra in the Diophantine text and that regarding the influence of that text on the birth and development of algebra.

With regard to the second point, there should be no doubt that, from the very first Arabic translators and commentators to the algebraists of the 16th and 17th centuries, Diophantus has been read in an algebraic key. Indeed, we can say that the interest in his work developed precisely in concomitance with the development of algebraic techniques. What Schappacher calls the 'first renaissance of Diophantus' takes place in parallel with what is recognized as the first Arabic text of algebra, the *Al-jabr* by al-Khwārizmī. In effect, Qusṭā ibn Lūqā's translation of the first seven books of the *Arithmetica* can be dated to 870, just a few months after the publication of the volume by al-Khwārizmī. Thus, the translation is strongly influenced (in terminology and in symbolism) by the new algebra and it is in the same way that the Diophantine text is interpreted by later readers. Acerbi thus rightly underlines that we are dealing with an algebraic reading of Diophantus and not with a direct influence of the Greek mathematician on the formation of an algebraic language:

> *È Diofanto che è tradotto in linguaggio algebrico, non Diofanto che induce la rivoluzione algebrica.* [25]

> It is Diophantus who is translated into an algebraic language, not Diophantus who leads the algebraic revolution.

The 'second algebraic renaissance', that carried out by the algebraists of the 16th and 17th centuries, has similar characteristics. As is known, after a first attempt at editing Diophantus' *Arithmetica* (by Giuseppe Auria at the beginning of the 16th century), the first real impact of Diophantus' work on the nascent algebraic culture of the West came from Bombelli's *Algebra* of 1572 [repr. 1966]. Here it is interesting to read what the algebraist from Bologna says in his own words:

> *Essendosi ritrovato un'opera greca di questa disciplina [l'Algebra] nella libraria di Nostro Signore in Vaticano, composta da un certo Diofante Alessandrino, Autor Greco, il quale fu a' tempo di Antonin Pio, & havendomela fatta veder Messer Antonio Maria Pazzi Reggiano, pubblico lettore delle Matematiche in Roma, e giudicatolo con lui Autore assai intelligente de numeri (ancorchè non tratti de numeri irrationali, ma solo in lui si vede un perfetto ordine di operare) egli, & io, per arricchire il mondo di così fatta opera ci dessimo a tradurlo, e cinque libri (delli sette che sono) tradutti ne habbiamo; lo restante non havendo potuto finire per gli travagli avenuti all'uno, e all'altro, e in detta opera habbiamo ritrovato, ch'egli assai volte cita gli Autori indiani, col che mi ha fatto conoscere, che questa disciplina appo gl'indiani prima fu, che à gli Arabi.* [Bombelli 1966, 8–9]

> There being found a Greek work of this discipline [*scil.* algebra] in the library of Our Lord in the Vatican, composed by a certain Diophantus of Alexandria, a

> Greek author who lived in the time of Antoninus Pius, and having been shown it by Mr Antonio Maria Pazzi from Reggio Emilia, a public lecturer in mathematics in Rome, and along with him deeming the author to be quite knowledgeable about numbers (though he does not treat irrational numbers but still in him is seen a perfect order of working), he and I, to enrich the world with such a well-made work, set about translating it. Five books (of the seven that there are) we have translated; we have not been able to finish the remaining books owing to the troubles that have befallen both of us. In said work we found that he many times cites the Indian authors, by which he has made me know that this discipline was first known by the Indians before the Arabs.

I transcribe the entire quotation by Bombelli because, in spite of his disconcerting statement that he had seen that Diophantus 'many times cites the Indian authors', it appears to me to confirm Acerbi's statement: just as for Arabic mathematics, so too in the case of the West, the introduction of algebra preceded, and did not follow, the comprehension of Diophantine mathematics. However, it should also be said that all of the early algebraists found it entirely natural to 'read' Diophantus in light of the 'algebraic revolution'. That is particularly true for the reading of the *editio princeps* by Bachet de Mezirac (1621).

It can be stated without any doubt that the insistence of Bombelli and the early algebraists on the 'algebraic' reading of the Diophantine text was profoundly motivated by 'ideology'. Their aim in fact was to give a 'classical' pedigree to algebra, freeing it from the purely practical status that it had assumed since the work of the abacus masters. Diophantus was, like Euclid, Apollonius, and Archimedes, to be counted among the noble fathers of the new mathematics: the search to rediscover the thread connecting the new problems and the classical tradition of geometry would continue throughout the whole of the 17th century, identifying above all in the methods of 'geometric analysis' the direct antecedent of the modern integration of algebra and geometry. In this way would Viète, Descartes, Schooten, Newton, and great number of others express themselves. In more recent times, in a historiographical context no longer tied to the problems of active research, Frederick Zeuthen proposed reading both the *Conics* of Apollonius and book 2 of the *Elements* in terms of the so-called 'geometric algebra'. The heated debate about such questions has characterized a series of interventions by various historians. But we will not enter into that here.[5]

The reading of Diophantus' work and, above all, the reflections on the kind of problems proposed in his book is relevant also for modern number theory, which can be said to have originated in Fermat's study of Bachet's edition of Diophantus. Here, for example, is how André Weil introduces his admirable history of number theory:

> One might similarly try to record the date of birth of the modern theory of numbers; like the god Bacchus, however, it seems to have been twice-born. Its first birth must have have occurred at some point between 1621 and 1636, probably closer to the latter date. In 1621, the Greek text of Diophantus was published by Bachet, along with a useful Latin translation and an extensive commentary. It is not known when Fermat acquired a copy of this book…nor when he began to read it; but, by 1636,…he had not only read it carefully, but was already developing ideas of his own. [Weil 2007, 1–2]

Going back to the thread of Acerbi's statement, while we can reasonably affirm that 'it is Diophantus who is translated into an algebraic language, not Diophantus who leads the algebraic revolution', the relationship to number theory is much more complex, even if Weil's observation does not at all imply (nor did Weil intend such an implication) that Diophantus was a precursor of Fermat, who read him with the eyes of a modern mathematician.

From a historiographical point of view, aside from the unquestionable importance of the influence of the reading of the *Arithmetica* on developments in algebra and number theory, two important questions remain:

(1) Into which tradition is the work of Diophantus to be inserted?
(2) What, independent of later readings, is the mathematical language of the *Arithmetica*?

With regard to the first question, it is well known that a considerable number of historians of mathematics have emphasized a presumed connection with Babylonian mathematics. Although it seems to me that such a connection—defended by historians of mathematics and mathematicians of the calibre of Neugebauer [1934, 245–259] and Van der Waerden[1954]—is based on clues that are too fragile, I believe that Acerbi's dismissal of the question is excessively perfunctory. (With regard to Neugebauer, he says that 'he

[5] On this question, see Saito 2004, 383–480.

saw connections that were non-existent but strategic for him in maintaining the pretext of paleo-Babylonian algebra' [14]). Instead, I find more cogent Acerbi's careful and in-depth examination of the studies of arithmetic on the part of Greek mathematicians, in particular Archytas, Nicomachus, and Iamblichus, regarding the insertion of different kinds of proportional means (arithmetic, geometric, harmonic, and so on) among given numbers. The examination of this rich mathematical tradition concludes, obviously, in an interesting and detailed account of what can be said, without fear of contradiction, to be 'the most substantial treatment of number theory in the Greek corpus previous to the *Arithmetica*', that is, of books 7–9 of Euclid's *Elements*.

Aside from the admirable (and very well known) theorems contained in these books (the algorithm of the greatest common divisor, the infinity of prime numbers, and so forth), Acerbi rightly turns his attention to some of the basic definitions, whose complete interpretation also requires attention to linguistic aspects: e.g., to that of a number ('a multitude composed of units') and, above all, that of a part ('the less of the greater when it measures the greater' [Heath 1956, 2. 277]), that is, a divisor, to those of plane and solid numbers (respectively, 'two numbers having multiplied one another' and 'three numbers having multiplied one another' [Heath 1956, 2. 278]). It should be noted, as Acerbi does, that plane and solid numbers are not mutually exclusive: e.g., 30 is both a plane number $(10\times3)$ and a solid number $(2 \times 3 \times 5)$.

Again in the context of this classification of number, book 9 of the *Elements* concludes with some theorems regarding perfect numbers, that is, numbers whose sum of their parts (i.e, their divisors excluding the number itself) is equal to the number itself, e.g., 6 $(= 1 + 2 + 3)$, 28 $(= 1 + 2 + 4 + 7 + 14)$, and so forth. The last proposition, *Elem.* 9.36, which Acerbi [36] rightly defines as 'the true τέλος [aim] of the arithmetic books', is, in Heath's translation:

> If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect. [Heath 1956, 3.421]

In other words, if a prime number is the sum of powers of 2, that number multiplied by the last power in the sequence from 1, will give a perfect number. In the case of 28, for example, 7 is a prime number and the sum of $1 + 2 + 4$, powers of 2; thus, 28 ($= 4 \times 7$) will be a perfect number.

This proposition, like all in the arithmetic books, is important for the study of polygonal numbers. In fact, written in another way, since the sum of $n$ powers of 2 (beginning with 0) is $2^n - 1$, if $2^n - 1 = M$ is prime, proposition 9.36 can be rewritten:

$$2^{n-1} \times (2^n - 1) = \frac{M \times (M + 1)}{2} \text{ is a perfect number.}$$

Hence, the perfect numbers identified by Euclid[6] are polygonal numbers (to be precise, triangular numbers). There is also another profound connection between book 9 of the *Elements* and Diophantus' *De polygonis numeris*: Euclid's book makes systematic use of the properties of the sums of elements in geometric progression (such as those of the powers of 2), while the Diophantine text studies polygonal numbers that are the sum of elements in arithmetic progression.

Thus, we have arrived at the central theme of Acerbi's book, the critical edition and study of Diophantus' *De polygonis numeris*. As already said, that study, although it constitutes the focus of the author's attention, occupies only a small part of the volume.

Let us, however, proceed in order: the text of *De polygonis numeris* is inserted, at least in part, in all of the Greek manuscripts that contain the books of the *Arithmetica*. But, in spite of that, numerous scholars, starting with Tannery, have cast doubt on its paternity. Personally, I do not believe that there is good reason to doubt the traditional attribution, though I note with Rashed and Houzel [2013, 4], that 'La différence, non seulement d'arithmétique mais de style, ne peut pas que surprendre' ('the difference, not only of arithmetic but of style, cannot help but surprise').

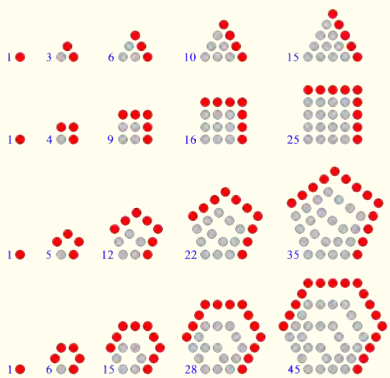
[6] The history of *Elem.* 9.36 is very interesting. The primes of type $M$ are called Mersenne primes and their study offers many open questions. While Euler proved that the even perfect numbers are necessarily of the type identified by Euclid, it is not actually known if there are any odd perfect numbers; it is only known that if they exist, they must be extremely large. It is worth noting that Nicomachus had already stated (but proved incorrectly) that there do not exist any odd perfect numbers.

Returning to the contents of the text, it must be said that the polygonal numbers are given by the sum of the first consecutive elements of an arithmetic progression always beginning with 1. They are divided into:

- triangular numbers, when the common difference is 1, that is, the progression of integers that gives rise to the numbers 1, 3 (= 1 + 2), 6 (= 1 + 2 + 3), 10 (= 1 + 2 + 3 + 4), 15 (= 1 + 2 + 3 + 4 + 5),...
- square numbers, when the common difference is 2: thus, 1 $(= 1^2)$, 4 $(= 1 + 3 = 2^2)$, 9 $(= 1 + 3 + 5 = 3^2)$, 16 $(= 1 + 3 + 5 + 7 = 4^2)$, 25 $(= 1 + 3 + 5 + 7 + 9 = 5^2)$,...
- pentagonal numbers, when the common difference is 3: thus, 1, 5 (= 1 + 4), 12 (= 1 + 4 + 7),... and
- hexagonal numbers, when the common difference is 4: thus, 1, 6 (= 1 + 5), 15 (= 1 + 5 + 9),...

and so forth.

The study of the properties of polygonal numbers goes back to the tradition of the Pythagorean school but there is no doubt that the first proofs that have come down to us are precisely those of Diophantus. The term ‘polygonal’ attributed to these numbers comes from the fact that the arithmetic progressions can be obtained by arranging the numbers according to geometric shapes and bordering them with gnomons. For greater clarity, I provide the figures of the first series:[7]




Triangular Numbers

Square Numbers

Pentagonal Numbers

Hexagonal Numbers

[7] The images are taken from the website of wikipedia.

In his definition, Diophantus says (as translated by Acerbi):

> *Ciascuno dei numeri aumentati di un'unità a partire dalla triade è poligonale primo a partire dall'unità ed ha tanti angoli quanta è la molteplicità delle unità in esso e suo lato è il numero di seguito all'unità, il 2. E sarà il 3 triangolare, il 4 quadrato, il 5 pentagonale e questo di seguito.*[8] [41]
>
> Each of the numbers augmented by a unit beginning from the triad is polygonal prime beginning from unity and has as many angles as is the multiplicity of the unit in it and its side is the number following unity, 2. And 3 will be triangular, 4 square, 5 pentagonal, and so on.

Thus, every number (except for 2) is polygonal in different ways: it is polygonal prime when it is the first of the progression after unity. In any given polygonal number, the number of angles is equal to the polygonal prime that generates it and a side equal to 2 (the side of every prime polygonal) augmented by a unit for each step of the progression beginning with the prime polygonal. Thus 15 is a triangular number (three angles and side 5) but also hexagonal (six angles and side 3) and 15-angles (15 angles and side 2).

Rather than go into the details of the individual propositions, which are in any case described with great accuracy by Acerbi, I will come to that which—even according to Diophantus himself—constitutes the goal of the entire booklet. This is to explore how to recognize a number as polygonal and determine its side. Naturally, since every number is a polygon, one must specify which polygon is being dealt with, that is, to identify the number of it angles or its 'species'. Thus, the central problem addressed in Diophantus' work is: given an integer, can we establish if it is polygonal of a certain angle?

To this end, the first point highlighted by Acerbi is that of putting into a mathematically significant form the intuitive concept of 'figurate' or polygonal number, as illustrated on page 272. This objective is achieved, as said earlier, by defining the $k-mo$ polygonal number of $P$ angles as the sum of $k+1$ terms of the arithmetic progression that begins with 1 and from the difference between elements $r = P - 2$. In this way, the definition is put into a form that is mathematically clear. Diophantus' answer is that a

8 Heath's translation is actually too synthetic and neglects various aspects of the Diophantine definition: 'All numbers from 3 upwards in order are polygonal, containing as many angles as they have units, e.g., 3, 4, 5 etc.' [Heath 1910, 247].

polygonal number $N$ must be such that $8N(P-2)+(P-4)^2$ is a square. This is condition (a).

Bachet de Mezirac had already noted that, while it is true that condition (a) is necessary, it is not sufficient, giving as examples the fact that $8\times 2(5-2)+(5-4)^2=49$ is a square but that 2 is not a pentagonal number, and that $8\times 4(7-2)+(7-4)^2=169=13^2$ is a square but 4 is not a heptagonal number. Acerbi rightly observes that these counterexamples cannot be considered valid because the solutions where $N<P$ are 'obviously' to be discarded. (I remind the reader that by definition $P$ is the smallest element different from 1 in the progression,) In any case, the fact remains that the sufficiency of condition (a), even with the obvious added condition $N>P$, was not proved by Diophantus.

The other objection, this too present in Bachet, concerns the fact that the Diophantine definition mentioned above is not applied to either the triangular numbers (for which $P-4=-1$) or to the square numbers (for which $P-4=0$). Acerbi justly criticizes the answer given by Hultsch [1876] to this problem, namely, that Diophantus was capable of manipulating negative numbers and zero, an idea that is unanimously rejected by historians today. Instead, Acerbi's convincing argument is simply that the formula given by Diophantus is a generalization of the known cases of triangular and square numbers, which thus do not enter into the typology presented by the proposition. As is easily verified, the triangular numbers $T$ satisfy the relation $8T+1=8T(P-2)+1=$ square, while the square numbers Q satisfy the relation $8Q(P-2)=$ square. Both cases satisfy what Diophantus said in the introduction to *De polygonis numeris*:

> Any polygonal multiplied into a certain number depending on the number of its angles, with the addition of to the product of a certain square also depending on the number of the angles, turned out to be a square. [Heath 1910, 247]

This characterizing property is proposed as the generalization of square numbers. As can be seen, the Diophantine proposition responds perfectly to what is proposed.[9]

[9] It is precisely in this context, that is, in comparing his proposition with what was proved by Hypsicles, a mathematician who lived in the second century BC, that Diophantus cites Hypsicles, a tenuous datum in dating Diophantus, if one assumes that he wrote the *De polygonis numeris*.

The last propositions of the *De polygonis numeris* teach how to determine the side of a given a number of a given species, and how to determine a number given its side (and species).

Proposition 5 is mutilated. It says: 'Given a number, find in how many ways it can be polygonal'. The proposition, proved by Bachet in a way that is unconnected with what has come down to us from the text of Diophantus, is hypothetically reconstructed by Wertheim [1897, 121–126] and Heath [1910, 256]. Acerbi offers an interesting appendix that reprises what he himself has already been published [2011, 548–560] and offers a new, more convincing reconstruction, which I will not repeat here.

As said earlier, the *De polygonis numeris* is a work that does not receive much attention from historians and number theorists. Acerbi's reconstruction and commentary effectively brings this work to the notice of scholars and situates it in a more harmonious way within the Diophantine corpus.

On the other hand, as is known, this treatise also played an important role in Fermat's thinking. Without going into these aspects, which are in any case rather well known, I should like to mention another marginal note in Bachet's edition, yet again incomplete because of the 'lack of space'. I am talking about observation 46, in reference to the proposition already cited that makes it possible to determine the polygon from a given side and *vice versa*. Fermat writes, extending the concept of figurate number to arbitrary dimensions:

> I will set out here, without demonstration, a very beautiful and very remarkable proposition that I have discovered: In the natural progression starting at unity, the product of an arbitrary number times its immediate successor makes double the triangle of the first number. If the multiplier is the triangle of the number immediately following, we have three times the pyramid of the first number. If it is the pyramid of the number immediately following, we have the quadruple of the 'triangulotriangulaire' of the first number, and so on indefinitely, following a uniform and general rule. I deem that a more beautiful or general theorem regarding numbers could not be stated. I have neither the time nor the space to put the demonstration in this margin.[10] [trans. from http://science.larouchepac.com/fermat]

Another famous observation by Fermat regarding polygonal numbers is no. 18, a remark on Bachet's comment on Diophantus, *Arith.* 4.31 containing

[10] Pengelley 2013 describes an interesting use of the square in a large didactic project.

an unproved proposition stating that every number, if it is not a square, is the sum of at most four squares. Fermat observes in the margin:

> What's more, there is a very beautiful and altogether general question which I have been the first to discover. Every number is: either triangular, or the sum of 2 or 3 triangles; Either square, or the sum of 2, 3, or 4 squares; Either pentagonal, or the sum of 2, 3, 4, or 5 pentagons; and so on indefinitely, whether it be of hexagons, heptagons, or any polygons; this marvelous proposition can be enunciated generally by means of the number of angles. I cannot here give the demonstration, which depends on numerous and abstruse mysteries of the Science of Numbers. I have the intention of dedicating an entire Book to this subject and thus, in this part of Arithmetic, I intend to make shocking progress beyond the formerly known limits.[11] [trans. from http://science.larouchepac.com/fermat]

On the other hand, Bachet had dedicated to precisely this work two long appendices, which Fermat read and assiduously commented on. That he was an attentive reader of not only the results but also of the methods used by Diophantus in the *De polygonis numeris* is also confirmed by Michael Mahoney, Fermat's biographer, regarding the importance of Bachet's reflection on the use of the sums of progressions:

> In the realm of summation formulas for the powers of integers Fermat's Archimedean model…could offer little inspiration.… Instead Fermat found his inspiration in Bachet's appendix to Diophantus' treatise *On Polygonal Numbers*. [Mahoney 1994, 229]

It is precisely on the methodological aspects of this work that Acerbi dwells.

Regarding the methods of solution used by Diophantus, by means of a linguistic and mathematical-philological argument, Acerbi distinguishes Diophantus' methods of analysis from the classical ones set forth by Pappus. The differences are significant: Diophantus omits a genuine synthesis in his argumentation and the approach of the *Arithmetica* appears to work more by reduction than to be an actual analysis. (The method of reduction consists in transforming an expression into an equivalent expression until it obviously assumes the form of what is hypothesized in the statement of the proposition, e.g., the form of a square.) Another method that Acerbi identifies, especially in the *De polygonis numeris*, is what he calls a 'chain of givens'. This is an argument of the type 'if A is a given (by hypothesis), then B will

[11] Fermat's proof is not known; the theorem was proved by Cauchy in 1813.

also be given and, thus, also C and, thus,...' until what is sought is obtained as a given. Here again, this concerns variants of the analytical method and Acerbi does well to examine them all with such care.

His examination focuses on problems akin to those indicated at the beginning about geometrical algebra:

(1) Is it possible to identify a method used by Diophantus in the solution of problems?
(2) Or is it a case of inventions that are efficacious in each individual situation but unconnected to each other?

A very clear and peremptory answer, shared by many scholars, is that given by Hankel:

> Of more general comprehensive methods there is in our author no trace discoverable: every question requires a quite special method, which often will not serve even for the most closely allied problems. It is on that account difficult for a modern mathematician even after studying 100 Diophantine solutions to solve the 101st problem; and if we have made the attempt, and after some vain endeavours read Diophantus' own solution, we shall be astonished to see how suddenly he leaves the broad high-road, dashes into a side-path and with a quick turn reaches the goal, often enough a goal with reaching which we should not be content; we expect to have to climb a toilsome path, but to be rewarded at the end by an extensive view; instead of which our guide leads by narrow, strange, but smooth ways to a small eminence; he has finished! He lacks the calm and concentrated energy for a deep plunge into a single important problem; and in this way the reader also hurries with inward unrest from problem to problem as in a game of riddles, without being able to enjoy the individual one. Diophantus dazzles more than he delights. [Hankel 1874, 159; trans. in Heath 1910, 54–55]

The search for these general methods led to the beginning of the 'algebraic' reading of Diophantus' text, a reading of which I spoke earlier and which often turns out to be historically insufficient.

Recently, following the rediscovery of Diophantine books in Arabic, the discussion has been taken up again in terms that I find interesting but which are not mentioned in Acerbi's book. I will permit myself to mention them here.

One interpretation worth noting is related to the reading of Diophantus' text not so much through the filigree of algebra as much as through that of modern number theory and, thus, of algebraic geometry. Perhaps the first to advance this line of interpretation was Isabella Bašmakova [1974],[12] who published a study on Diophantus and Diophantine analysis. According to this point of view, number theory in Diophantus is traced to aspects relating to the study of algebraic curves and to the search for their rational points. For example, the search for Pythagorean triads comes down to the search for the rational points of the circumference $x^2 + y^2 = z^2$.

With regard to this attempt at interpretation, Schappacher's stand is drastic:

> Certain historians of mathematics try to surpass the mathematicians in blending modern inspiration with Diophantus' alleged thoughts. The worst example of this thoughtless tendency is given by the Russian historian of mathematics Bašmakova in her book on Diophantus. [Schappacher 2005, 27–28]

This judgment is much too harsh. Much more interesting, I believe, is the nuanced judgment of Houzel and Rashed:

> *Quoique «forcée» et ne pouvant pas prétendre au titre d'historique, cette lecture d'I. G. Bašmakova a le mérite d'expliquer les procédures réglées en usage dans les* Arithmétiques*, procédures qui laissent supposer un ordre précis qu'aucune autre lecture n'était en mesure d'expliciter.* [Rashed and Houzel 2013, 43]

> Although 'forced' and unable to claim to be history, this reading of I. G. Bašmakova has the merit of explaining the procedures set in use in the *Arithmetica*, procedures that suggest a specific order that no other reading has been able to explain.

In any case, Bašmakova's reading paved new roads for interpretation.

Thus, Weil, who guarded against attributing to Diophantus a role as precursor of modern algebraic geometry but nevertheless read the Greek text with the eyes and language of a 20th-century mathematician, deepened Bašmakova's insight. Here I will limit myself to citing this significant passage:

> *On ne peut pas manquer d'être frappé déjà chez Diophante, de la fréquence avec laquelle reviennent les équations qui définissent de courbes de genre 0 ou 1, et du fait que ce sont toujours les mêmes méthodes que Diophante mette en ouvre pour les résoudre.* [Weil 1981, 398]

[12] This is actually a translation of the original Russian text of two years earlier. See Bašmakova 1997 for an English translation.

> One cannot fail to be struck already in Diophantus, by the frequency with which equations that define curves of genus 0 or 1 turn up, and the fact that it is always the same methods that Diophantus puts to work for the solution.

This point of view concerning Diophantus' work has been amply developed by Houzel and Rashed in many studies of Arabic mathematics. The two scholars write:

> *Si donc nous refusons de lire dans les* Arithmétiques *les notions de la géométrie algébrique et celle de l'Analyse diophantienne contemporaine, nous proposons en revanche de conserver ces moyens théoriques, mai au seul titre d'instrument, comme outil théorique qui permet d'identifier les méthodes et aussi de mieux connaitre les liens entre les 280 problèmes traités par Diophante et d'éclaircir la structure de son livre.* [Rashed and Houzel 2013, 43–44]
>
> So while we refuse to read into the *Arithmetica* concepts of algebraic geometry and contemporary Diophantine analysis, we propose instead to retain these theoretical means but only as an instrument, as a theoretical tool that permits us to identify methods as well as to understand better the connections among the 280 problems addressed by Diophantus and to clarify the structure of his book.

This appears to me to be a useful comparative methodology.

A different attempt in the effort to find a methodological thread in the work of Diophantus, comes from another well-known scholar of the subject, Jean Christianidis. This is a very different line of thinking: not a key for reading but for the exposition of Diophantus' actual intentions in writing the text. Here is how Christianidis presents the method used by the Greek mathematician:

> We are now in a position to present Diophantus' general method of arithmetical problem solving....
>
> The canon of Diophantus for solving arithmetical problems:
>
> (1) Invention–transfer of the problem (in its instantiated version) to the framework of the "arithmetical theory", i.e., transformation of the problem into an equation;
> (2) Disposition—transformation of the equation into its final form, and finding the unknown number;
> (3) Computation of the numbers sought; and
> (4) Test proof. [Christianidis 2007, 300]

He adds:

> Diophantus' intention in the *Arithmetica* is not to present a theory for solving algebraic equations. His goal…was to elaborate a canon on the basis of which several arithmetical problems could be treated and to demonstrate how this canon should be used in practice. [Christianidis 2007, 303]

As we see, there are a great number of different interpretations of the presence of a unifying method in the work of Diophantus, which are sometimes in clear contrast with each other and sometimes mutually complementary. Acerbi's account is particularly tied to the philological aspects and merits attentive study.

I have attempted to give an idea of a field of study that has interested mathematicians and historians of mathematics for at least 1000 years, and that appears not to have exhausted its potential. Before closing, I should mention an ulterior field of interest for these problems, that of didactics. I refer here to a recent publication by Anne Michel-Pajus who examines Acerbi's work on polygonal numbers [2011] in detail, focusing especially on the propositions pertaining to the determination of the side of a polygonal number of a given species and *vice versa*, as an example of argumentation in which we go from the formula to the algorithm and conversely:

> We chose this text, because Diophantus gives three presentations for the same mathematical property: one with a 'rhetorical formula', and one with an algorithm, then the inverse algorithm…. Teachers are used to going from the algorithm to the formula, as the formula is more familiar to them (and to current students). We see here how Diophantus extracts an algorithm from a formula. This is a commonplace task in elementary mathematics. [Michel-Pajus 2012, 376]

It seems to me that these diverse uses also show how a careful analysis of the text can clarify methodological questions of no small importance.

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