Taming the Unknown: A History of Algebra from Antiquity to the Early Twentieth Century by Victor J. Katz and Karen Hunger Parshall

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Writing about the history of algebra is fraught with difficulties and even dangers. Scholars have disagreed about the definitions of even the basic terms (What is algebra? And what is history?) and opponents have carried on vigorous and sometimes ill-tempered debates, not just about the validity of one another's work but also about one another's competence.

The most difficult issue to resolve is the nature of algebra itself. Part of the problem is that the meaning of the word 'algebra' has a changed significantly over the last 1,200 years. It begins, in the work of al-Khwarizmi (about AD 825), as the name of a single operation (the restoration of a subtracted quantity) carried out during the process of solving for an unknown quantity. But, by 1600, it became the name for that whole process and, because of Viète's *In artem analyticam isagoge* (published in 1591), it came to include the idea of using symbols (ordinary letters) to represent both known and unknown quantities. But algebra continued to evolve well into the 20th century. In an article tellingly entitled 'The Beginnings of Algebraic Thought in the Seventeenth Century', Michael Mahoney [1971] offered the following definition (partially quoted in the present book [4–5]; I have italicized the missing parts):

> *First, then,* what should be understood as the 'algebraic mode of thought'? It has three main characteristics: first, this mode of thought is characterized by the use of an operative symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates. Second, precisely because of the central role of combinatory operations, the algebraic mode of thought deals with mathematical relations rather than objects. *Even when certain relations become themselves objects, say the set of a group morphisms, one seeks the*

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> *relations that link these new objects. The subject of modern algebra is the structures defined by relations, and thereby one may note as a corollary that the algebraic mode of thought rests more on a logic of relations than on a logic of predicates.* Third, the algebraic mode of thought is free of ontological commitment. *Existence depends on consistent definition within a given axiom system, and mutually compatible mathematical structures live in peaceful co-existence within mathematics as a whole.* In particular, this mode of thought is free of the intuitive ontology of the physical world.

Although this definition sought to bring clarity to a famous debate about whether there is, for example, algebra hidden in book 2 of Euclid's *Elements*, in the context of this review it actually highlights several sources of difficulty. First, in modern times, the word 'algebra' has fractured into two distinct meanings. On the one hand, there is the algebra familiar to high-school and college-level pre-calculus students, which 'simply' involves operating on symbols and equations with the aim of finding unknown quantities. This corresponds roughly to the first of Mahoney's criteria and, to a certain extent (but there could be room for argument here), it also engages with his third criterion. On the other hand, there is the 'modern algebra' (to use Mahoney's phrase, even if he may not have intended a separate meaning) which is a product of 19th- and 20th-century interest in generalization and structure; and this algebra certainly satisfies all three of Mahoney's criteria. Mahoney used his definition to argue that there could not be any algebra in book 2 of Euclid's *Elements*. But from the point of view of the present book, it means that a historian really has two histories to write: one for algebra and one for modern algebra.

However, Mahoney's definition raises another issue for us. It is a definition of a mode of thought rather than of (say) a use of symbolic manipulation and so it could suggest the possibility that this mode of thought was present before the invention of the symbolic tools we now associate with it. So maybe al-Khwarizmi was using algebraic thought even though his quantities, equations, and operations were purely verbal (that is, 'rhetorical', in the usual terminology of the history of algebra) rather than symbolic. And maybe there was algebraic thought present even before the use of the word algebra in the calculations of ancient Babylonians or the arguments in book 2 of Euclid's *Elements*.

The idea that something might be present before it is named, or before it is even noticed, is a recurring theme in the present book and it highlights a difference of approach between historians and mathematicians when studying the history mathematics. Ivor Grattan-Guinness [2004] has described these approaches as 'history' and 'heritage' where, roughly speaking, the history approach tries to describe what happened in terms of the culture of the time, while the heritage approach tends to ask what modern mathematics has inherited from the chosen episode, person, or culture. However, the difference is not always clear, since even the historian may need to reach across the centuries to render historical mathematics in a form which is intelligible to modern readers, and since different people are sensitive to different levels of such intervention.

For example, I cringe a bit when S. Ahmad and R. Rashed [1972] talk about the method that al-Samaw'al used in the 12th century 'to find the root of a square element of the ring':

$$Q[x] + Q[\frac{1}{x}]'.$$

For me, the ring structure is more or less irrelevant to al-Samaw'al's calculations. Rings are a late 19th-century abstraction of all the different mathematical systems in which you can add, subtract, and multiply and in which (roughly speaking) these operations behave like ordinary addition, subtraction, and multiplication of real numbers. But, in fact, rings include systems where multiplication does not necessarily satisfy the commutative law; so the use of the word 'ring' can conjure up alien and anachronistic associations for a modern reader.

On the other hand, and again this is just my opinion, there are occasions when the judicious use of symbolic algebra can illuminate mathematics done 100s or even 1000s of years before Viète had the idea of assigning letters to unknown or known quantities. For example, what we now call linear problems in several unknowns arose 100s of years before the invention of symbolic algebra; and a truly authentic account of them would be purely verbal, with the problems stated in words, as in the following example from the *Liber Abbaci* written by Leonardo of Pisa (or Fibonacci) in 1202 [Sigler 2002, 317].

> Two men with *denari* find a purse with *denari* in it.
>
> The first says to the second, 'If I were to have the *denari* from the purse along with those I already have, I would have three times as much as you.'

> To which the other replies, 'And if I were to have the *denari* from the purse along with my *denari*, I would have four times as much as you.'
>
> How much does each man have and how much is in the purse?

The solution process, too, would be spelled out in (possibly several pages of) wordy explanation. In this context, most modern readers would probably like to see the problem stated symbolically, if only because the symbolism strips away the 'irrelevant' information and reveals the structure of the problem in a much briefer and more familiar (and so, easier to grasp) form. Thus, readers feel that they know and understand the nature of the problem being solved. For example, the above problem could be represented in terms of two symbolic equations:

$$a + p = 3b$$
$$b + p = 4a$$

There is little doubt that modern readers lose something by this simplification but it is a way of enticing them to step out of their own culture and make the effort to understand the historical culture. There are also risks with such simplification though. If Leonardo is able to solve this problem (and others like it, involving up to five men) can we conclude that Leonardo could solve (some) systems of linear equations in up to six unknowns, even though no one would write down such equations for another 400 years or so? Using symbolic representation could lead the reader to think the answer is 'Yes!' But others may feel it is nonsensical to claim that someone could solve a problem that they could not even formulate.

Mathematicians do seem to be particularly prone to what historians might call anachronisms but what mathematicians might see as new ideas in old settings. Part of the problem here is that, as the present book shows, such recognition is often a crucial part of the way in which research mathematicians actually do mathematics. The extent to which the mathematician claims to see the new in the old does vary. For example, the old may just be a source of raw material, as when Lagrange carries out a 'detailed review of the existing solution methods for third- and fourth-degree equations' [295] on his way to discovering the role of permutations in the solvability of such equations [296–298]. Or the old may be a source of inspiration, as when Sylvester and Cayley find 'the germ of a whole new theory that they would call invariant theory' while reading Boole's work on the effect of linear transformations on higher degree 'forms' [353]. But sometimes the mathematician claims that everything was already there in the past. Perhaps the most spectacular example of this is Viète's claim that his new 'analytic art' (effectively, symbolic algebra) was nothing but ancient Greek analysis dressed in modern clothes [236]. It is not clear to what extent he thought that this was clever marketing in a society that revered classical Greek culture, as opposed to a recognition of his key ideas embedded in the ancient texts. But, in a similar vein, we have the 'plausible but unprovable assumption', this time made by A. Weil [1984, 170] on behalf of Fermat, that Fermat's original proofs in number theory 'could not have differed much' from those obtained by Euler about a century later. This behavior puts the historian of mathematics in a bit of a bind. Do you stick to the printed evidence or should you allow yourself to be swayed by the mathematical expert who says things like, 'these two quite different-looking things are really exactly the same' and 'this person must have been thinking such-and-such because that is the way you think when you know this subject as well as they did'?

To a certain extent, how you approach these issues depends on your intended audience. The present book targets readers with a college major in mathematics [3], educated laypeople if you like, not research specialists in the history of mathematics. The authors have judged, correctly I think, that such readers are more likely to appreciate a story told mostly in their 'native tongue', in this case symbolic algebra. For example, ancient Babylonian problems which we can interpret as quadratic equations are represented that way [24], despite the fact that the algebraic symbolism would not be invented for another 3,000 years or more. But the reader is still given a flavor of the 'foreign language' by way of a verbal account of the solution process as given on the original clay tablets, along with Høyrup's conjectured cut-and-paste geometric construction which explains where that process might have originated. Similarly, there is copious use of the phrase 'what today would be called' as a way of connecting quite foreign-looking historical accounts with the intended reader's modern viewpoint. In most cases, this is accompanied by a sketch, at least, of what the calculations (or other thought processes) looked like to the original participants. Thus, the readers see, for example, the relative clumsiness of the first explanations of many ideas compared with the slick and polished presentations in modern textbooks.

The book falls into three main sections. Chapters 1 to 8 deal with the prehistory of algebra, what the authors call 'algebraic thought' despite Mahoney's definition mentioned earlier, up to the advent of symbolic algebra. In these chapters, the authors adopt Euler's definition of algebra [6] as 'the science which teaches how to determine unknown quantities by means of those that are known'. This part of the story begins in the earliest history of mathematics with algebraic thinking being found in ancient Egyptian, Babylonian, Indian, and Chinese problem-solving, as well as in the traditional Arabic birthplace of al-Khwarizmi's 'algebra'. This section also includes special mention, in chapter 3, of the 'geometric algebra' that has caused so much dissent in the past. Here the approach is even-handed, acknowledging that Euclid's book 2 is geometric in intent (the history) but accepting too that this same material was used by later writers (the heritage) to justify their algebraic calculations.[1]

Chapters 9 and 10 discuss the invention of symbolic algebra and how it was used to solve polynomial equations and to support the invention of coordinate geometry. This part of the story begins with the amazing tale of how cubic and quartic polynomial equations were solved by Scipione del Ferro, Nicolo Tartaglia, and Ludovico Ferrari without the help of symbolic algebra. Again, many of the painful details are rendered intelligible to us lesser mortals by using that algebra. The first historical use of algebra that we learn about is the invention by Fermat and Descartes of coordinate geometry [ch. 10].

Chapters 11 to 14 deal with the evolution of what we might call high-school algebra into modern algebra, with its concern for generalization and structure. This evolution begins with the slow dawning of realization that higher degree polynomials might not always be solvable. It is ironic that symbolic algebra, which Viète had touted as a universal problem solver [237], should be the tool used to reveal that some problems cannot be solved. Without algebra's ability to condense verbal calculations and strip away all inessential distractions, it is difficult to imagine how anyone could ever have dealt

[1] It is interesting, incidentally, that although the authors reference Nesselmann's classic paper of 1842 which broke down the development of algebra into three stages (rhetorical, syncopated, and symbolic), they do not seem to mention it in their text. Assuming this neglect is deliberate, I agree completely, as it is hard to see Nesselmann's syncopated step actually occurring as a second step. After all, its first appearance was supposed to be in Diophantus, long before the 'rhetorical' writings of al-Khwarizmi and those of his Arabic successors who were familiar with Diophantus.

with such complications as Lagrange's permutations of polynomial roots [295–297] or how anyone could have found the equivalent of the new algebraic tools for investigating solvability (such as what we now call groups [300–303] and fields [310–312]). The final chapters show how the recurrence of these 'structures', especially groups and fields, but also what we now call matrices, vectors and linear transformations, all led mathematicians to see value first in abstraction and then in axiomatization. The book concludes with an account of how van der Waerden's two-volume book *Moderne Algebra* [1930–1931] was based on the lectures of Emmy Noether and Emil Artin, and how it came to popularize what Mahoney understood as algebra.

The authors have, I think, pitched their writing perfectly for their intended audience. The broad outline of the story is expressed in clear prose, combined with a judicious use of that other 'native tongue' of the college mathematics graduate, symbolic algebra. If the reader is willing to make a further effort, then there is sufficient detail in other forms (often paraphrases of the wordy originals) to give an experience somewhat closer to reading the original historical documents. And for the really keen reader, there is an extensive bibliography presenting the more detailed historical research that has been carried out, particularly in the last 30 years. You could base a really nice third-year course on this book.

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