The Meaning of « ἐνὶ ὀνόματι » in the *Sectio canonis*

by

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Abstract

A new interpretation is proposed of the crucial expression «ἑνὶ ὀνόματι» ("in one name") as applied to ratios of the musical concords in the preface of the *Sectio canonis* ascribed to Euclid. A link is also established with the name of one of the irrational lines introduced by Euclid in *Elements* 10. Past interpretations of the expression are discussed and shown to be inadequate.

About the Author

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1. An interpretative problem

The introduction of the *Sectio canonis* ascribed to Euclid ends by setting a correspondence between concordant notes and certain kinds of numerical ratios:

> πάντα δὲ τὰ ἐκ μορίων συγκείμενα ἀριθμοῦ
> λόγῳ λέγεται πρὸς ἄλληλα, ὥστε καὶ τοὺς φθόγγους
> M158.20 ἀναγκαῖον ἐν ἀριθμοῦ λόγῳ λέγεσθαι πρὸς ἀλλήλους·
> τῶν δὲ ἀριθμῶν οἱ μὲν ἐν πολλαπλασίῳ λόγῳ λέγονται,
> οἱ δὲ ἐν ἐπιμορίῳ, οἱ δὲ ἐν ἐπιμερεῖ,[1] ὥστε καὶ τοὺς
> φθόγγους ἀναγκαῖον ἐν τοῖς τοιούτοις λόγοις λέγεσθαι
> πρὸς ἀλλήλους. τούτων δὲ οἱ μὲν πολλαπλάσιοι καὶ ἐπι-
> μόριοι ἑνὶ ὀνόματι λέγονται πρὸς ἀλλήλους.[2]
> Γινώσκομεν δὲ καὶ τῶν φθόγγων τοὺς μὲν συμφώ-
> νους ὄντας, τοὺς δὲ διαφώνους, καὶ τοὺς μὲν συμφώνους
> μίαν κρᾶσιν τὴν ἐξ ἀμφοῖν ποιοῦντας, τοὺς δὲ διαφώ-

[1] In a multiple ratio, the greater term is a multiple of the lesser. In an epimoric ratio, the excess of the greater term over the lesser term is a part (i.e., a divisor) of the latter. In an epimeric ratio, this excess is "parts" of the lesser term: "parts" of a given number is any number less than the given one that is not a part of it. The current characterizations of these relations as fractions, as we shall see below, is simply misleading. I shall use the denominations "epimoric" and "epimeric" in place of the more common "superparticular" and "superpartient".

[2] A look at the particles in this sentence suggests that something has gone wrong. The initial « δέ » is mildly adversative, as is the « δέ » at the beginning of the sentence opening the second paragraph. This is in line with the careful disposition of the *cola* in the whole introduction: independent, principal clauses are always introduced by conjunctive « δέ », and inside them the subclauses in contraposition are regularly marked by the canonical correlative « μέν…δέ ». Moreover, every « μέν » is answered by a « δέ ». The only exception is the « μέν » in the underlined sentence [lines 24–25]: a subsequent clause such as « οἱ δὲ ἐπιμερεῖς οὔ » (whereas epimeric do not) is surely missing. I regard the correction as certain, given the strictly analogous structure of the immediately following sentence. Nothing in the interpretation that I shall develop depends on this textual detail, however.

νους οὔ. τούτων οὕτως ἐχόντων εἰκὸς[3] τοὺς συμφώνους
M160.1 φθόγγους, ἐπειδὴ μίαν τὴν ἐξ ἀμφοῖν ποιοῦνται κρᾶσιν τῆς φωνῆς, εἶναι τῶν ἐν ἑνὶ ὀνόματι πρὸς ἀλλήλους λεγομένων ἀριθμῶν,[4] ἤτοι πολλαπλασίους ὄντας ἢ ἐπιμορίους. [Jan 1895, 149.8–24; Menge 1916, 158.18–160.4; Barbera 1991, 114.15–116.11]

> Now all things that are composed of parts are compared to each other in a ratio of number, so that notes too must be compared to each other in a ratio of number. Some numbers are compared in a multiple ratio, some in an epimoric ratio, and some in an epimeric ratio, so that notes must also be compared to each other in these kinds of ratio. And of these, the multiple and the epimoric are compared to each other in a single name.
>
> Among notes we also recognize some as concordant, others as discordant, the concordant making a single blend out of the two, whereas the discordant do not. In view of this, it is to be expected that the concordant notes, since they make a single blend of sound out of the two, are among those numbers which are compared to each other in a single name, being either multiple or epimoric. [Barker 1984–1989, 2.192–193, modified]

Two entangled problems in the argument have attracted the attention of commentators. The first is the status of the so-called "principle of consonance", namely, that concordant notes must be represented either by multiple or epimoric ratios.[5] I shall not discuss this issue here. The second is the meaning of the expression «(ἐν) ἑνὶ ὀνόματι» (in a single name): this is the characterization, admittedly rather cryptic, of multiple or epimoric ratios that allows setting any of them in correspondence with notes that make a single blend.[6]

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3 εἰκός: notice the determination of likelihood in a place where in the first paragraph one finds two occurrences of a determination of necessity (ἀναγκαῖον). I would link this feature to a perceptibly less firm status of the assumed correspondence between notes and numbers. Compare the more precise statement occurring on the second line of the first paragraph: «τοὺς φθόγγους ἀναγκαῖον ἐν ἀριθμοῦ λόγῳ λέγεσθαι πρὸς ἀλλήλους».

4 The *variatio* «(ἐν) ἑνὶ ὀνόματι» between lines 158.25 and 160.2 is very likely a scribal *lapsus*, even if it is not clear whether the mistake is a haplography or a dittography.

5 The problem lies in the fact that the introduction of the *Sectio* apparently expresses the principle as a sufficient condition only, whereas in *Sectio* 11 the converse is explicitly applied.

6 As the second underlined clause confirms [lines 160.2–3], the demonstrative «τούτων» in the line 158.24 refers to numbers and not to classes of ratios or of notes. As a consequence, what is qualified by the "single name" clause is each single ratio, not

2. Ancient commentators

The ancient commentators did not address the question of the "single name". Neither Porphyry nor Boethius, when reporting the introduction of the *Sectio*,[7] remains faithful to the received text.[8] Porphyry skips altogether the portion of the argument beginning with the first sentence underlined in the text. Boethius provides a paraphrase of the entire final part but does not render the occurrences of "single name" in his abridged version. This could mean either that they thought the meaning of "single name" unimportant or obvious or that they were too puzzled about it to point out the problem or to survey earlier (if any existed) interpretations.

3. Current interpretations

The interpretations of the expression "single name", which I shall call "current", derive from a proposal first elaborated in a paper by L. Laloy [1900], a proposal which has been rediscovered a few times since then. Laloy introduces his central claim when he explains «(ἐν) ἑνὶ ὀνόματι» by remarking that in ordinary usage ancient Greek has single words to denote each particular multiple and epimoric ratio only. As he observes, terms denoting epimoric ratios, being more complex in principle than terms for multiple ratios, are formed according to a fixed rule so that any such ratio can be easily named. But the ordinary language of ancient Greece does not offer similar terms for the other kinds of ratios. The occurrence of single words designating epimeric ratios in Nicomachus, *Intro. arith.* 1.20–21—at any rate much later a work than the *Sectio*—is restricted to a fairly technical context. Indeed, the very exposition in Nicomachus, Laloy says, suggests that he is really handling very uncommon terms or maybe even coining them.[9]

whole classes of multiple or epimoric ratios (which would be a truism). The correspondence set forth in the introduction of the *Sectio* requires in fact that one single ratio be related to one single concord, since any of the latter makes a single blend. Of course, any single epimoric or multiple ratio stands for a whole class of equivalent ratios. For simplicity, I shall refer to each class as if it were one single ratio.

[7] At Düring 1932, 90.7–23, and *De inst. mus* 4.1–2 [Friedlein 1867, 301.12–302.2], respectively. It should be noted that Porphyry does not mention the *Sectio* in his quote, whereas he expressly refers to it at Düring 1932, 98.19, when reporting an extensive initial segment of the deductive part of the same treatise.

[8] We may exclude the possibility that the occurrences of "single name" are later additions to the introduction of the *Sectio*, since they are integral parts of the argument.

[9] The shorter account by Theon of Smyrna [Hiller 1878, 78.6–22] employs only two- or many-word phrases to name epimeric ratios; elsewhere [109.15–110.18], Theon

As for the the omission of the phrase «(ἐν) ἑνὶ ὀνόματι» in Boethius' abridged translation, Laloy has this explanation:

> Le fait de langage auquel il est fait allusion est propre au grec: les mots *sesquiquartus*, *sesquiquintus*,...sont des mots savants forgés pour les besoins d'un ouvrage d'arithmétique: ils ne peuvent être invoqués comme des preuves. Euclide, au contraire, trouvait toutes formées, dans sa langue, des locutions usuelles qui sont à ses yeux des témoins irrécusables. [Laloy 1900, 239]

Scholars after Laloy have either sided with him or rediscovered his interpretation: so, for example,

P. Tannery 1904, 445,
C. E. Ruelle 1906, 319,[10]
E. Lippmann 1964, 154,
W. Burkert 1972, 383n63,[11]
A. Barker 1981, 2–3; 1984–1989, 2.192–193 nn6–8, and
A. Barbera 1991, 55–58.[12]

In her Italian translation of the *Sectio*, L. Zanoncelli [1990, 63–64] further qualifies Laloy's interpretation in asserting that the reference is to the single numeral appearing in the designation of a (multiple or) epimoric ratio,[13] such as «ἐπίτριτος» and so on.[14] Unfortunately, besides regularly formed terms for epimeric ratios such as, e.g., «ἐπιδίτριτος»,[15] which contains two

introduces one-word denominations that are different from Nicomachus'. This means that the terminology was not fixed but does not entail that the terms were of recent coinage. Theon and Nicomachus were contemporaries.

10 In fact simply relying on Tannery's authority.

11 Burkert does not argue his claim but adduces (pseudo-)Aristotle, *Prob*. 19.34 and 41 as *loci paralleli*. Yet only the latter has a reliable text and, though it can be compared more properly to some propositions in the *Sectio*, it does not bear on the principles set forth in the introduction [see the translation in Barker 1984–1989, 2.95–96].

12 Barbera apparently came to know of Laloy's paper after a communication by A. Kárpáti.

13 The name of an epimoric ratio is always the name of the ratio in lowest terms identical to it. As an epimoric ratio in lowest terms is of the form $(n + 1) : n$, only one "number" (in Greek sense, hence excluding unity) has to be named. This is already pointed out by Theon of Smyrna, [Hiller 1878, 77.5–7]. A similar remark, this time pointing to the single number appearing in the anthyphairetic expression of an epimoric ratio, is found in Fowler 1999, 141.

14 This is "one third more" and corresponds to $4/3$ in least terms.

15 This is "two thirds more" and corresponds to $5/3$ in lowest terms.

numerals, there are alternative names of the same ratios containing one numeral, in this case «ἐπιδιμερής». Therefore, Zanoncelli has not isolated a characterization that can serve as a criterion for singling out multiple and epimoric ratios.

Alternative interpretations take different routes. Assuming that some precise word is referred to in the introduction of the *Sectio*, proposals for such a single word have been advanced by a number of scholars. Jan suggests "potior" (more powerful)—

> Porphyri...nomen illud commune affert, cum potiores (κρείττους) dicit has duas rationes: Euclides ea brevitate et dicendi inopia haec agit, ut excerpta potius dicas quam ipsa verba hominis sagacissimi. [Jan 1895, 118][16]

> Porphyry provides such a common name when he says that these two ratios are "more powerful". Euclid treats these things so succinctly and in so few words, that you would regard them more as excerpts than the words themselves of this most brilliant man.

—and Mathiesen puts forward "consonant" [Mathiesen 1975, 254n12]. But these alternatives are defended on the basis of an incorrect reading of a text by Porphyry, who asserts only that multiple and epimoric ratios are more powerful than epimeric in the same way as consonant and melodic notes are more powerful than dissonant ones, and concludes that one should thereby "fit" («ἐφαρμοστέον») multiple and epimoric ratios to consonant notes, epimeric ratios to dissonant notes.[17] Porphyry's explanation is in fact nothing but a slight restatement of the very passage in Ptolemy's *Harmonica* 1.5 on which he is commenting [see Düring 1930, 11.8–20]. Both Porphyry and Ptolemy are far from claiming that either "consonant" or worse yet "more powerful" is the single name referred to in the *Sectio*: neither mentions the "name" and Ptolemy even ascribes the whole argument expounded in 1.5 to the "Pythagoreans".[18]

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[16] The absence of the "name" induced Jan to conjecture the existence of a richer version of the argument in another Euclidean treatise.

[17]
> Τῶν οὖν ἀνίσων λόγων οἱ μὲν πολλαπλάσιοι καὶ οἱ ἐπιμόριοι κρείττους τῶν ἐπιμερῶν, τῶν δ’ ἀνισοτόνων κρείττους οἱ ἐμμελεῖς καὶ οἱ σύμφωνοι τῶν ἀσυμφώνων. ἐφαρμοστέον ἄρα τοὺς ἐπιμορίους καὶ πολλαπλασίους λόγους τοῖς συμφώνοις, τοὺς δ’ ἐπιμερεῖς τοῖς ἀσυμφώνοις. [Düring 1930, 98.3–6]

[18] Just after that, Ptolemy quickly summarizes formalized arguments—he asserts that they conclude γραμμικώτερον (more rigorously)—which are an abridgment of *Sectio* props. 11, 10, 12; and he refers to the results established in props. 3, 6, 13, and 16.

More interesting is the *mathematical* explanation provided by Ptolemy of the asserted superiority of multiple and epimoric ratios to epimeric ratios. The basic assumption, Ptolemy says, was that

> οἱ μὲν ἴσοι τῶν ἀριθμῶν παραβληθήσονται τοῖς ἰσοτόνοις φθόγ-
> D11.10 γοις, οἱ δὲ ἄνισοι τοῖς ἀνισοτόνοις, τοὐντεῦθεν ἐπάγουσιν, ὅτι καθάπερ τῶν ἀνισοτόνων φθόγγων δύο ἐστὶν εἴδη πρὸς ἄλληλα τὰ πρῶτα, τό τε τῶν συμφώνων καὶ τῶν διαφώνων, καὶ κάλλιον τὸ τῶν συμφώνων, οὕτως καὶ τῶν ἀνίσων ἀριθμῶν δύο γίνονται πρῶται διαφοραὶ λόγων, μία μὲν ἡ τῶν λεγομένων ἐπιμερῶν καὶ ὡς ἀριθμὸς πρὸς ἀριθμόν,[19] ἑτέρα δὲ ἡ τῶν ἐπιμορίων τε καὶ πολλαπλασίων, ἀμείνων[20] καὶ αὕτη τῆς ἐκείνων κατὰ τὴν ἁπλότητα τῆς παραβολῆς,[21] ὅτι μέρος ἐστὶν ἁπλοῦν ἐν αὐτῇ τῶν μὲν ἐπιμορίων ἡ ὑπεροχή, τῶν δὲ πολλαπλασίων τὸ ἔλαττον τοῦ μείζονος.[22]
> [Düring 1930, 11.9–17]
>
> Equal numbers should be associated with equal-toned notes, and unequal numbers with unequal-toned; and from this they argue that just as there are two primary classes of unequal-toned notes, that of the concords and that of the discords, and that of the concords is finer, so there are also two primary distinct classes of ratio between unequal numbers, one being that of what are called "epimeric"or "number to number" ratios, the other being that of the epimorics

Accordingly, Porphyry's transcription of a substantial part of the *Sectio*, with explicit reference to its title and mention of Euclid as the author [Düring 1932, 98.19], is but an expansion of Ptolemy's sketchy proofs. On the issue, see the discussions in Barker 1994 and Barker 2000, 54–73. Barker assigns the Pythagorean argument to Archytas.

[19] For the latter denomination, see Plato, *Tim.* 36b. It might be surmised that the former is a more recent and more technical term, the latter an archaic one. Alternatively, we might have here simply a quotation from Plato without technical implications. The Platonic expression is given a wrong explanation in Theon, *Exp.* [Hiller 1878, 80.7–14]: Theon asserts that the phrase singles out ratios different from those he has just described, not realizing that his own classification (which included multiple-epimoric and -epimeric ratios besides the usual ones) is exhaustive.

[20] Porphyry varies the term to «κρείττους» using the plural to refer to the ratios.

[21] Barker's translation [1984–1989, 2.285] has "comparison" (at the beginning of the quotation, the passive future of the related verb is rightly translated "associated"). But «παραβολή» (application) is here employed as a technical term coming from the theory of the application of areas: an area is applied to a straight line when the area is transformed into a rectangle having the straight line as one of its sides. In numerical context, «παραβολή» simply means "division" or the resulting "quotient", and the corresponding verb («παραβάλλω») means "to divide": for the verb, see, e.g., Acerbi and Vitrac 2014, 159n36 and Tannery 1893–1895, 2.278 *sub voce*.

[22] Porphyry's paraphrasis [Düring 1932, 98.7–13] simply makes the argument clumsier.

> and multiples; and of these the latter is better than the former on account of the simplicity of the application, since in this class the difference, in the case of epimorics, is a simple part, while in the multiples the lesser is a simple part of the greater. [Barker 1984–1989, 2.284–285, slightly modified]

Ptolemy's argument appears to imply that the "single name" is warranted not by language but by a mathematical property shared by both multiple and epimoric ratios.

A quick reading of this passage and of the paraphrase in Porphyry may underlie arguments that the "name" is "more powerful" or "consonant". But note that Ptolemy (or his "Pythagorean" sources) reverses the order of the main inference found in the *Sectio* by making the classification of ratios depend on that of concords. Moreover, since concords are defined on aesthetic grounds just at the end of the preceding chapter of the *Harmonica* [see Düring 1930, 10.25–28], the same semantic field is naturally at one's disposal to denote ratios too. For this reason, Ptolemy qualifies multiple and epimoric ratios as "better" («ἀμείνων») than epimeric ratios. Still, we should not mistake such a judgment as grounds for identifying the "name" in the *Sectio*.

A. C. Bowen [1991, 176–182] argues at length for "concordant" as the name, using an approach that is different from any of the others just described. The core of the argument is that the predicates "multiple" and "epimoric" can be applied directly to notes since in the *Sectio* phenomenal musical sounds (i.e., sounds as described by intervals related by certain ratios) and objective musical sounds (i.e., sounds analyzed as series of consecutive motions) are identified. This reading precludes from the very outset any reference to numbers and ratios as such, and the problem of the "single name" really evaporates since what we actually hear are the ratios. Solving a problem by dissolving it is an elegant way to cope with *aporias* but we shall presently see that a satisfactory answer can be given within the traditional interpretative framework, in which notes and ratios are kept distinct.

The interpretations of the "single name" phrase proposed by most modern scholars stress a linguistic feature, although one linked to a mathematical property. The basic weakness of all such proposals lies in the fact that in ancient Greek it was far from impossible to form one-word descriptions of epimeric ratios. On the contrary, ancient Greek is more than capable of doing this, as we have seen. Moreover, it is disputable that Nicomachus' denominations of epimeric ratios were his own invention: after all, he does not claim it as his own and the names are formed in accordance with a rule that is a natural extension of the one for epimoric ratios. Nor is it a problem that the first occurrences of names for particular epimeric ratios are found first in Nicomachus and in Theon of Smyrna, considering what has survived of ancient number theory.[23]

4. The concept of name (ὄνομα)

There is a very specific property of multiple and epimoric ratios making them suitable to be ranged under the extension of the same description. It is a mathematical and not a linguistic feature, even if the two aspects have a large overlap *because* the names of such ratios are in general built up looking at some mathematical property.

A first point, showing that the context is less specifically linguistic than usually believed, can be made concerning the verb «λέγειν». It occurs six times in the introduction of the *Sectio*, in the passive and possibly qualified by «πρὸς ἀλλήλους» (to each other). The first four occurrences refer to notes or numbers that are in relation to each other by means of a ratio; the latter two refer to numbers that are in relation to each other "in a single name". The parallelism of the two verbal constructions is obvious. Translations of «λέγεσθαι» such as "to be spoken of"[24] load the expression with philosophical overtones and unduly stress the linguistic connotation of the verb. The most proper translation of «λέγεσθαι πρὸς ἀλλήλους» is "to be compared to each other" in all its occurrences here.

This is in line with one of the current meanings of «λέγω» [see Liddell, Scott, and Jones 1968, *sub voce* (B).I] and comparable to the usage in the preface to Archimedes, *De lineis spiralibus*:[25]

> τᾶν ἀνισᾶν
> γραμμᾶν καὶ τῶν ἀνίσων χωρίων τὰν ὑπεροχάν, ᾆ
> ὑπερέχει τὸ μεῖζον τοῦ ἐλάσσονος, αὐτὰν ἑαυτᾷ συν-
> H12.10 τιθεμέναν δυνατὸν εἶμεν παντὸς ὑπερίσχειν τοῦ προ-
> τεθέντος τῶν ποτ' ἄλλαλα λεγομένων.
> [Heiberg 1910–1915, 2.12.7–11]

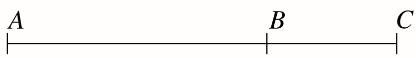
23 Theon's account [Hiller 1878, 74.15–75.25] might suggest that older classifications knew only of multiple and epimoric as independently defined classes of ratios; but the closing of his exposition [1878, 75.17–21] seems to imply that Theon suggests this possibility only by way of rhetorical expedience.

24 E.g., Barker 1984–1989, 2.192–193.

25 A similar formulation is also found in the fifth assumption at the beginning of *De sph. et cyl.* 1.

> Of unequal lines and of unequal areas, the excess by which the greater exceeds the lesser, if added to itself, can exceed any proposed ‹magnitude› among those that can be compared to each other.

What is more, even if the term «ὄνομα» has an obviously prominent linguistic connotation, it also carries a peculiar and well-defined mathematical meaning. To see this, notice first that in general a ratio between two numbers can be represented as a divided line as follows:[26]



Let us suppose, without loss of generality, that AB is the greater segment of AC. It may happen that BC measures exactly AB. But by definition

$AC : BC$ is multiple whenever BC measures AC (and hence AB) exactly.
$AC : AB$ is epimoric whenever BC measures AB (and hence AC) exactly.[27]

Therefore, $AC : BC$ is multiple if and only if $AC : AB$ is epimoric; and this happens if and only if BC measures exactly AB. As a consequence, multiple and epimoric ratios are built upon a single reference number BC, let us call it a single "name", in the strong sense that BC is the common measure of all the numbers at issue in such ratios.

No other ratios share this property. Such a fundamental characterization of multiple and epimoric ratios is completely obscured by their usual representation as ratios of the form $mn : m$ and $(mn + m) : mn$, respectively, or, if reduced to lowest terms as is usually and even more misleadingly done, $n : 1$ and $(n+1) : n$. In particular, what is lost is the key role played by the notion of "part" of a number in the ancient definitions of multiple and epimoric ratios.

The characterization just expounded is purely mathematical; for two reasons, it does not coincide with the one that was expounded in the preceding section and is an integral part of the "current" interpretation. First, in the latter, the "single name" of multiple and epimoric ratios derives from the (name of the) number corresponding to the greater segment AB, that is, number n in the ratios $n : 1$ and $(n + 1) : n$. But in the interpretation

[26] Nothing in the following argument depends on the possibility of representing numbers by line segments.

[27] Cf. p. 39 n2 above. Ancient definitions can be found, e.g., in Theon of Smyrna, *Exp.* [Hiller 1878, 76.8–14 (multiple), 76.21–77.2 (epimoric)]. Less perspicuous definitions are in Nicomachus, *Intro. arith.* 1.18–19. Of course, the definitions state necessary and sufficient conditions.

just presented, the "single *name*" is the *number itself* (and not its name)[28] corresponding to the *lesser* segment BC (namely, number m in the ratios $mn : m$ and $(mn + m) : mn$). If we like, when dealing with a ratio, our focus can be either on the common measure of the terms of the ratio or on the pair of numbers by which one must multiply such a common measure to generate the terms themselves.[29] My proposal assumes the former point of view; the "current" interpretation surveyed above assumes the latter.

Second, since what is referred to in the ordinary names of multiple and epimoric ratios is the number corresponding to the greater segment, the present interpretation does not require that there be a predicate which answers to "concordant" and which singles out multiple and epimoric ratios.[30]

In my view, the phrase "single name" in the introduction of the *Sectio* should be taken as a reference to a "single *name*", i.e., to a mathematical object. Thus, I would render the sense of

> τούτων δὲ οἱ μὲν πολλαπλάσιοι καὶ ἐπιμόριοι ἑνὶ ὀνόματι λέγονται πρὸς ἀλλήλους [Menge 1916, 158.24–25]

by

> The multiple and epimoric numbers are compared to each other [*scil.* in ratio to each other] with respect to a single reference-number.

All of this would be just a refinement and a completion of Ptolemy's explanation κατὰ τὴν ἁπλότητα τῆς παραβολῆς (because of the simplicity of the application) [see p. 44, above], were it not for a lucky accident that permits adding some historical flesh that squares rather well with the proposed interpretation. This is the use of the term "name" for a mathematical object in the theory of irrational lines.[31]

[28] I shall henceforth use "name" in italics to denote a mathematical object denominated in this way.

[29] Of course, the two multiples are the terms of the ratio expressed in lowest terms.

[30] Unless the predicate is simply taken to be "having a single *name*" (i.e., being described by a single reference number). If we assume that the ratios are in lowest terms, we might even hold that there is in fact a common predicate to all multiple and epimoric ratios, namely, "having the unit as their *name*".

[31] I have not been able to find any relevance to our subject in the notion of "homonymous" parts and numbers at work in *Elem.* 7.37–39 and in Diophantus' *Arithmetica*. Apollonius' usage of «ὁμώνυμος» as reported by Pappus in *Coll.* 2.1–16 deserves a more careful assessment but appears to be irrelevant to our purposes.

In book 10 of the *Elements*, a binomial—in Greek, ἐκ δύο ὀνομάτων (from two *names*)—is a line formed by composition of two expressible lines that are commensurable in power only.[32] It is first defined at *Elem*. 10.36 and its *names* are expressly mentioned dozens of times in the rest of book 10. In a diagram analogous to the one set out above, 10.36 amounts to saying that a line *AC* is a binomial if it is obtained by composing two expressible straight lines *AB* and *BC* such that *AB* is incommensurable with *BC* but the squares on them are measured by a common area.

In a testimony whose reliability is controversial, however, Pappus, on the authority of Eudemus, assigns a seminal role to Theaetetus, who is reported to have introduced and named the three basic kinds of irrational lines (medial, binomial, and *apotome*), linking them to the three basic means (geometric, arithmetic, and harmonic respectively).[33] At 968b19–20, the Peripatetic tract *De lineis insecabilibus* mentions the binomial line as well as the *apotome*.[34] It is, therefore, almost certain that the denomination "binomial" was introduced before the composition of *Elem*. 10. Moreover, the lines from which an *apotome* is obtained by subtraction are expressly called its *names* in *Elem*. 10.112–114[35] and such *names* of an *apotome* are set in one-to-one correspondence with the *names* of a suitable binomial. This suggests that

32 An expressible line is any straight line set out as a reference-line or any line commensurable in power with it. Two lines are commensurable in power when the squares on them are commensurable. Lines commensurable in power are said to be "commensurable in power only" when they are not commensurable [*Elem*. 10.def.2]. On the notion of "expressible line", see also p. 52 n39, below.

33 Junge and Thomson 1930, 63: see also 138, where Eudemus is not mentioned. The authenticity of book 1 of Pappus' *Commentary* is doubtful: see Vitrac 1990–2001, 3.417–21. Pre-Euclidean interest in the theory of irrationals is of course attested in Plato's *Theaetetus*.

34 This small treatise is a product of the Peripatetic school. A work with the same title is included also in the list of Theophrastus' writings: see, e.g., Diogenes Laertius, *Vitae philos*. 5.42. Therefore, it is reasonable to assume that it was composed before the *Elements*.

35 The definition of an *apotome* in 10.73 is exactly symmetrical to the one of a binomial in 10.36: an *apotome* is a line formed by subtraction of two expressible lines that are commensurable in power only.

the *names* had a more widespread application than the one that the extant sources attest and that they lasted well beyond Euclid's times: since *Elem.* 10.112–114 are absent in the Arabo-Latin tradition, we may infer that they were introduced later into the text, very likely after Apollonius and certainly before Pappus, who read them.[36]

An even later tradition, which surfaces in the Theonine manuscripts and in the medieval Greco-Latin translation of the *Elements*, designates the segments from which other irrational lines are formed as *names*. This happens in the enunciations of *Elem.* 10.43–47, e.g., where a corrector in the unique pre-Theonine ms. Vat. gr. 190 has put the same qualification in the text of prop. 10.46 as well [see Heiberg and Stamatis 1969–1977, vol. 3 *in app. ad locos*].

As the two lines composing the binomial are called its *names*, one is naturally led to assume that the existence of some well-defined and basic mathematical object called *name* should precede the choice of such a denomination as "from two *names*". But then, what was that *name*?

To clarify the point, it may be useful to refer briefly to Aristotle, *Meta.* 10.1, where he lists examples of things for which it is necessary to set out more than one reference-measure. The last items are «καὶ ἡ διάμετρος δυσὶ μετρεῖται καὶ ἡ πλεύρα καὶ τὰ μεγέθη πάντα» (both the diagonal and the side are measured by two <reference-measures> as well as all magnitudes) [*Meta.* 1053a17–18]. Surprisingly enough, commentators since Alexander have been at a loss in explaining such a transparent sentence.[37]

Very simply, all Aristotle says is that since side and diagonal (of a square) are incommensurable, by definition there is no common measure to them

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[36] But it is likely that the *names* for the *apotome* were introduced to mimic the attested Euclidean usage for the binomial, not as a reference to a longstanding tradition harking back to earlier investigations.

[37] Alexander of Aphrodisias:

> For if <the diagonal> is measured, say, by a finger, the finger is twofold: the essence and the form of the finger and this <finger> here itself measuring it; and similarly also the side is measured by two since it is a magnitude. [Hayduck 1891, 610.4–6]

Aquinas:

> Similiter etiam est diameter circuli vel quadrati, et etiam latus quadrati: et quaelibet magnitudo mensuratur duobus: non enim invenitur quantitas ignota nisi per duas quantitates notas. [Cathala 1935, liber 10, lectio 2, §1951, 561b]

and, hence, to measure both of them one has to set out two independent reference-measures.[38] The generalization to all magnitudes is straightforward when they are geometrical and simply a matter of analogy when they are not. Nor should the syntax of the sentence bewilder us [Ross 1924, 283]: when a clause has two subjects, referring the verb (hence put in the singular) to the first subject and then adding the second subject paratactically is not an unknown pattern in Greek prose [cf., e.g., Smyth 1920, §966]. The Aristotelian allusion entails that setting out two different reference-measures in the field of irrational lines was a matter of course. Aristotle calls each of them the μέτρον but this was clearly a most generic denomination, dictated by the very subject of the second part of *Meta*. 10.1.

Let us return to the binomial. The two segments that compound such lines are incommensurable. Thus, it follows that two reference-lines are needed to measure them. My hypothesis is that the two *names* in the denomination of the binomial refer exactly to this feature.

H. Bonitz:

> hoc videtur significare, et rationem quae diagonalem inter et latus intercedit, et cuiuslibet planae figurae magnitudinem non definiri una linea mensurata, sed duabus mensuratis et mensurae numeris inter se multiplicatis. [Bonitz 1849, 418]

W. D. Ross:

> the diagonal is conceived as consisting of two parts, a part equal to the side, and a part which represents its excess over the side. [Ross 1924, 283]

In following Göbel, Ross deems the mention of “the side” as “the gloss of an overzealous copyist”.

T. L. Heath:

> the relative lengths of the diagonal and the side can be approximated to by forming the successive approximations to $\sqrt{2}$ in accordance with Theon of Smyrna’s rule: these are $7/5$, $17/12$, $41/29$, etc. If therefore we took the side to be 1, we could say that the diagonal was one of these fractions, so that two numbers (one divided by the other) are required to measure it. [Heath 1949, 218–219]

Of course, Heath is bound to accept Ross’ excision of “the side”. Only in Burkert 1972, 462n74 does one find a correct assessment of the passage. However, Burkert refers quite misleadingly to the setting out of two reference-measures as an “expedient of practical geometry”.

[38] This was in fact a commonplace point: cf. Plato, *Parm*. 140b–c and the first scholium to *Elem*. 10 in Heiberg and Stamatis 1969–1977, 5.2 at 84.21–85.1.

Of course, one may well take the two segments themselves that compound the binomial as reference-lines; and in this sense the binomial may appropriately be said to be composed "from two *names*". All of this, however, is at variance with the introduction in *Elem*. 10 of a single ῥητή (expressible) line as a reference-line. Both *names* of a binomial are in fact expressible lines, even if they are commensurable in power only. As a consequence, one single ῥητή is needed as a reference to build up a binomial, though the ῥητή itself is not a common measure of the *names*. This shows that the use of « ῥητή » in *Elem*. 10 should not be taken as coming from the same developments that yielded the coinage of "from two *names*" for the binomial.

The "metrological" conception of the reference-line as an standard of measurement was the one in use in the pre-Euclidean theory of irrational lines. This can be argued on the basis of a series of testimonies [see Acerbi 2008], including the well-known passage at *Theaet*. 147d–148b containing Theodorus' lesson and Theaetetus' definitions of "lengths" and "powers", and a handful of Aristotelian texts. It should then come as no surprise if the introduction of the peculiar notion of "expressibility" that we find in book 10 were original with it (we should, of course, suppose that the *Sectio* draws on a much earlier tradition).[39] Since to build up a binomial just one expressible line is required while two *names* were apparently needed, the introduction of the former notion might well have been devised as a simplifying feature.

5. Conclusion

Can we connect the "names" of some irrational lines with the "single name" in the introduction of the *Sectio*? From the preceding discussion a unified view of the two notions emerges naturally. The *name* of multiple and epimoric ratios is the single number that is the common measure of the two terms of such ratios: this we can surmise on the basis of the passage from Ptolemy's *Harmonica*. On the other hand, the *names* in a binomial irrational line are the two incommensurable lines needed to measure the two segments from which the binomial itself is obtained by composition. The tradition reports that the term "name" was used to denote also the components of other irrational lines. A *name*, I surmise, was a reference-measure, both in a geometrical and in a number-theoretical context. Going beyond these remarks would be rash. However, the interpretation advanced here has at

[39] For a thorough discussion of the ancient debate concerning the notion of expressibility in *Elements* 10, see Vitrac 1990–2001, 3.43–51.

least the virtue of proposing a unified view of two hitherto unrelated objects in Greek mathematics denoted by the same *name*.

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